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OPTIMUM INSENSITIVITY OF THE DISCRETE-CONTINUOUS TRANSFORMATION

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C O N T E N T

SYMBOLS

$=$	equals; is equivalent to
$\neq$	does not equal
$\equiv$	identically equals to
$>$	greater than
$<$	less than
$\geq$	greater than or equal to
$\leq$	less than or equal to
$\hat{y}$	estimated value of variable y
$E\{\ldots\}$	operator stands for the mathematical expectation
$\underline{\underline{A}}, \underline{\underline{B}}, \underline{\underline{C}}$	matrices
$\underline{\underline{A}}^T, \underline{\underline{B}}^T, \underline{\underline{C}}^T$	transposed matrices
$\underline{x}, \underline{u}, \underline{v}$	column vectors
$\underline{x}^T, \underline{u}^T, \underline{v}^T$	row vectors
$\underline{\underline{O}}$	zero matrix
$\underline{O}$	zero vector
$\underline{\underline{I}}$	unit matrix
$\lvert\underline{\underline{A}}\rvert$	determinant of matrix $\underline{\underline{A}}$
$tr(\underline{\underline{A}})$	trace of matrix $\underline{\underline{A}}$
$\underline{\underline{A}}^{-1}$	inverse of matrix $\underline{\underline{A}}$
$\frac{d}{d\underline{x}}$	ordinary derivative column operator
$\frac{df(\underline{x})}{d\underline{x}}$	derivative or gradient of function $f(\underline{x})$
$\frac{d\,\underline{f}(\underline{x})}{d\,\underline{x}^T}$	Jacobian matrix, vector derivative of function $\underline{f}(\underline{x})$
LS	Least Squares Method

GLS	Generalized Least Squares Method
ML	Maximum Likelihood Method
IV	Instrumental Variable Method
SGLS	STEIGLITZ' Generalized Least Squares Method
MLG	Maximum Likelihood Method with noise model of Generalized structure

I. INTRODUCTION

1.1 The formulation of the task

The design of engineering technological processes or the optimization of operating systems always requires mathematical modelling of the system to be controlled. In the case of linear dynamic systems this can be achieved among others by experimental identification, within these, often by very efficient discrete identification methods discussed in this report. On the one hand, the direct utilization of the results of this report can be expected in this field. On the other, some branches of industry in Hungary have attained a level enabling their investment plans to consider in certain areas the introduction of computer process control. The computer process control requires, however, the first and basic step on which the decision level of the process control is built, viz. the process identification to be realized with suitable accuracy. This field can be considered the most important one for utilizing the results of this report.

The process identification beginning with the classical graphoanalytic methods up to modern computer-aided procedures has always been popular among the researchers. In the frame of this report conceptual terminological issues of identification, the calssification of the methods are not discussed. These topics had already been dealt with on a very high level by famous authors on several IFAC Symposia and Congresses [11], [27] and some books have also been published covering this subject. In this report the terminology of the international literature dealing with identification is used, the notations are the same as in the papers of ÅSTRÖM's school and for the teaching of control theory at the Technical University of Budapest [22-24]. As the terminology of the international literature has become fairly widespread, definitions, denominations are elaborated on only in the most important cases.

The object to be identified is therefore assumed to be a linear, dynamic plant with concentrated parameters, or this latter restriction is in certain cases eased by permitting serial deadtime. Here it is assumed that the input and output signals of the process can be recorded in every sampling time $\Delta T = h =$ constant. (The measurements are supposed to be coherent.) Fig. 1.1-1 shows the measuring situation for a single input-single output system, where $u(t)$ is the input signal, $y(t)$ the output signal, u_k and y_k denote their values sampled in the moments of time $t = k.\Delta t = k.h$. The argument t will denote (in the whole report) the continuous time, the subscript (now k) the discrete time of the model, as usual in the literature.

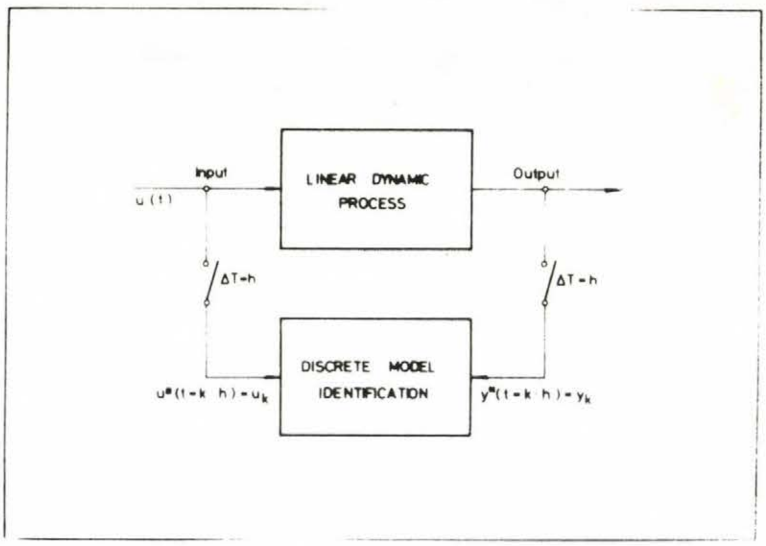

Fig. 1.1-1

The above measuring situation is suitable, of course, both for passive and active identifications. Thus, it contains, e.g. the identification based on the classical step responses, and at the same time the application of other, more modern methods. This type of approach has got in the literature the name of discrete identification which designation refers to the discreteness of the measuring circumstances on the one hand, and also to the discrete-time feature of the applicable models, on the other.

Due partly to the measurement data available generally in discrete form in the various measurement situations and partly (may be mainly) to the (often on-line) data processing and evaluation which could be performed, thanks to the spread of computers, by the digital computer, at the beginning of the 60-s the attention has been directed to the discrete system description, i.e. the discrete process models. Thereupon followed the quick development of theory and techniques of discrete identification. Perhaps Kalman [11] was the first to publish in this field, but the basically pilot activity was due to the group with Åström [10]. Even today the off-line method worked out by them gives the best results and the base for comparison with other modes of solution.

Since then, several other approaches have been suggested which, might have been advantageous under their special conditions, but they actually correspond to the special cases of the original idea [12]. The modern identification methods, therefore, resort nowadays almost exclusively to discrete techniques the reason of which is mainly that the large quantity of computations requires digital computer. The loading of the measurement data into the computer can be achieved only by sampling (often the data logger also supplies the data directly in this form) and accordingly the identification methods, too, perform the estimation of the parameters of discrete time models.

With respect to the practical application, it can be stated that the off-line discrete identification methods of single input single output systems already mean the everyday identification techniques in numerous research places, in several countries and the literature reported already of the accomplishment of a great number of industrial modelling problems. In order to determine the original continuous system, the discrete model obtained as a result of the identification has to transform to the descriptive forms of the equivalent continuous system..

We may state with certainty that as a consequence of the spread of discrete identification methods, nowadays - when standard programs, program packages are available to the users - not algorithmic issues or problems connected with computer programming are staying more in the foreground but research tasks surfaced in the course of practical applications. The majority of these new problems concern the optimization of the measuring, experimental circumstances of the input and output signal series to be used for discrete identification, usually in order to enable us to obtain as much information as possible about the process for the model building or to perform the parameter estimation of the identification more exactly. Among others such tasks arised the demand for the identification-oriented optimal input signal synthesis (excitation) or for the optimization of the applied sampling time. This latter is the subject of this report, viz. how the sampling period for the discrete identification has to be chosen to get the best possible results.

II. DETERMINATION OF THE CONTINUOUS TRANSFER FUNCTIONS FROM DISCRETE TRANSFER FUNCTIONS

As we already stated in the Introduction, the parametric identification of the linear, dynamic, time invariant systems (apart from other problems of structure searching and parameter estimation) can be performed in two steps. In the first one a discrete model is defined by the identification which model yields a system fitting well to the input and output signals of the process at the sampling moments. In the second step a linear continuous model equivalent with the obtained discrete model has to be defined.

This problem area was already dealt with by many authors started from different approaches: e.g. GOLDEN [33], SMITH [96], LATZEL [64], JEŽEK [54], LATZEL and WIEGAND [65], HAYKIN [41], HSIA [50], SINHA [94]. The task itself is often called z-s transformation. The discussion of these problems occurs not only with regard to system identification, but also in connection with system simulation (inverse problem) and the twofold problems are difficult to be treated in separation. Now we face identification viewpoints.

As to the identification process mentioned already several times, let us look at Fig. 2-1. In this case the continuous

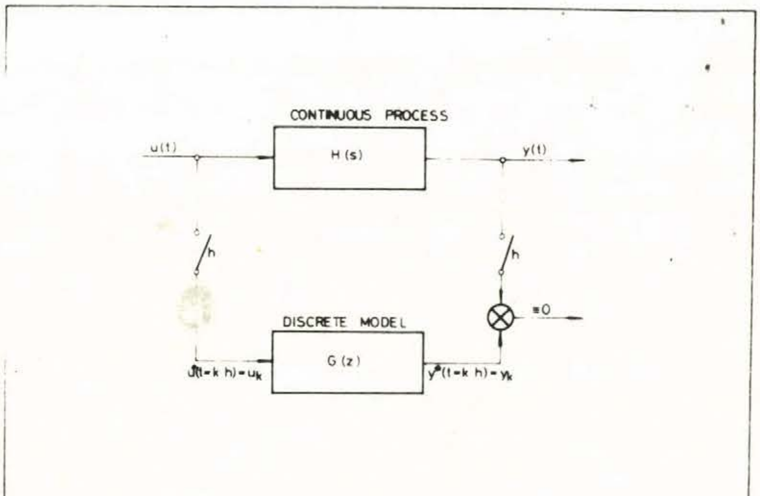

Fig. 2-1.

input and output signals u(t) and y(t) of the continuous system with the transfer function H(s) are sampled with the sampling time h. Our discrete model G(z) will be fitted to the discrete series $u^*(t=k.h)=u_k$ and $y^*(t=h.k)= y_k$. k index denotes therefore the serial number of the sampling period. Now for the sake of simplicity a noiseless case will be investigated and the condition of the good fitting of the discrete model is to yield a value identical with the output signal for the sampling instants, i.e. the difference signal must be zero.
Let G(z) be:

$$G(z) = \frac{B(z)}{A(z)} = \frac{b_o + b_1 z^{-1} + \dots + b_n z^{-n}}{1 + a_1 z^{-1} + \dots + a_n z^{-n}} , \qquad (2.1)$$

where z^{-1} is the so-called shift operator, i.e. $z^{-1}x_k = x_{k-1}$. The interpretation of this - generally speaking - discrete (and not pulse) transfer function can be seen from the following difference equation

$$y_k = B(z)u_k - \tilde{A}(z)y_k = \sum_{i=0}^{n} b_i u_{k-i} - \sum_{i=1}^{n} a_i y_{k-i} , \qquad (2.2)$$

where $\tilde{A}(z) = A(z)-1$. Therefore, the discrete transfer function G(z) makes a connection with an equivalent difference equation between the actual and preceding values of the output and input signal. Accordingly G(z) which can be obtained with identification procedures, describes the continuous system for the sampling instants in an equivalent way.

Now let us investigate this equivalence in depth. Fig. 2-2. demonstrates that the description forms G(z) and S^{-1} H(s)S are in fact equivalent. Here the ideal sampler is denoted by S and the ideal reconstructor or holding element by S^{-1}.

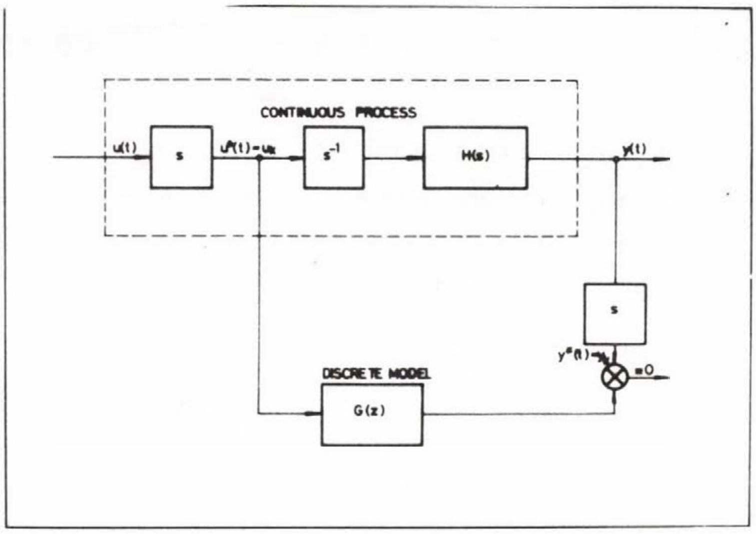

Fig. 2-2.

Thus S denotes the element realizing the sampling operator defined by LIFF [71].

The sampling operator is defined by the equation

$$F(s) = \mathcal{Z}\{ [\mathcal{L}^{-1}\{F(s)\}]^{*} \} = S\{F(s)\} , \tag{2.3}$$

where the z transformation is denoted by $\mathcal{Z}$, the Laplace transformation by $\mathcal{L}$, while $F(z)$ and $F(s)$ denote the corresponding transforms of the same signal.

From the equivalence shown in Fig. 2-2.

$$G(z) = S^{-1} H(s) S \tag{2.4}$$

follows the equation

$$U(z)G(z) = U(z)S^{-1}\ H(s)S = U(s)H(s)S\ , \tag{2.5}$$

where $U(s)$ is the Laplace transform of the input signal, and $U(z)$ the Z-transform of the same. On the basis of Eq. (2.5)

$$Z^{-1}\{U(z)G(z^{-1})\} = \left[\mathcal{L}^{-1}\{U(s)H(s)\}\right]^{*}, \tag{2.6}$$

hence by using the definition (2.3)

$$H(s) = \frac{1}{U(s)}\ S^{-1}\ \{U(z)G(z)\}. \tag{2.7}$$

In consequence, the definition of the equivalent continuous system is unambiguous only for a given input signal, i.e. for a given approximation of the ideal reconstructor, else not.

Consider a first order system where the identified discrete transfer function

$$G(z) = \frac{b_1 z^{-1}}{1 + a_1 z^{-1}} \tag{2.8}$$

and seek the continuous system equivalent for the step response in the form of

$$H(s) = \frac{\beta_1}{s+\alpha_1} = \frac{K}{1 + sT}\ . \tag{2.9}$$

On the basis of the Eqs. (2.6) - (2.7)

$$U(z)G(z) = Z\left\{\left[\mathcal{L}^{-1}\{U(s)H(s)\}\right]^{*}\right\} \tag{2.10}$$

i.e.

$$\frac{1}{1-z^{-1}}\;\frac{b_1 z^{-1}}{1+a_1 z^{-1}} = \frac{\beta_1}{\alpha_1}\;\frac{(1-e^{-\alpha_1 h})\,z^{-1}}{(1-z^{-1})(1-e^{-\alpha_1 h}z^{-1})} \tag{2.11}$$

because now $U(z) = 1/(1-z^{-1})$ and $U(s) = 1/s$.

By comparing the two sides of the equation, the relationship of the coefficients are:

$$\alpha_1 = -\frac{\ln(-a_1)}{h}\;;\quad \beta_1 = -\frac{b_1 \ln(-a_1)}{h(1+a_1)} = \frac{b_1 \alpha_1}{1+a_1} \tag{2.12}$$

or

$$K = \frac{\beta_1}{\alpha_1} = \frac{b_1}{1+a_1}\;;\quad T = \frac{1}{\alpha_1} = -\frac{h}{\ln(-a_1)}\;. \tag{2.13}$$

By using the results the transfer function

$$G(z) = \frac{b_o + b_1 z^{-1}}{1 + a_1 z^{-1}} = \frac{(b_1 - b_o a_1)\,z^{-1}}{1 + a_1 z^{-1}} + b_o \tag{2.14}$$

can be transformed easily to the step response equivalent continuous form

$$H(s) = \frac{\beta_o s + \beta_1}{s + \alpha_1} = K \frac{1 + T_1 s}{1 + Ts} , \qquad (2.15)$$

where

$$\alpha_1 = - \frac{\ln(-a_1)}{h}; \quad \beta_1 = \frac{b_o + b_1}{1 + a_1} \alpha_1; \quad \beta_o = b_o \qquad (2.16)$$

or

$$K = \frac{b_o + b_1}{1 + a_1} ; \quad T = - \frac{h}{\ln(-a_1)}; \quad T_1 = \frac{b_o}{K} T. \qquad (2.17)$$

By following the method of SMITH [96], the continuous transfer function step response equivalent with the identified second-order discrete transfer function can be calculated with the Eq. (2.10) giving the equivalence. Let be

$$G(z) = \frac{b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} , \qquad (2.18)$$

and the continuous system sought for

$$H(s) = \frac{\beta_1 s + \beta_2}{s^2 + \alpha_1 s + \alpha_2} = K \frac{1 + T_1 s}{1 + 2 \xi Ts + T^2 s^2} . \qquad (2.19)$$

The relations of the transformation resulting from the calculation are for complex roots:

$$\alpha_1 = -\frac{\ln(a_2)}{h} = 2\gamma_r \;;$$

$$\alpha_2 = \left(\frac{\ln(a_2)}{2h}\right)^2 + \left(\frac{\arc\cos\dfrac{-a_1}{2\sqrt{a_2}}}{h}\right)^2 = \gamma_r^2 + \gamma_i^2 \qquad (2.20)$$

where

$$\gamma_r = -\frac{\ln(a_2)}{2h}; \qquad \gamma_i = \frac{\arc\cos\left(\dfrac{-a_1}{2\sqrt{a_2}}\right)}{h} \qquad (2.21)$$

are auxiliary quantities. Coefficients in the numerator are:

$$\begin{bmatrix} \beta_1 \\ \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \\ \varphi_{21} & \varphi_{22} \end{bmatrix} \begin{bmatrix} b_1 \\ \\ b_2 \end{bmatrix}, \qquad (2.22)$$

where

$$\varphi_{11} = \frac{2\gamma_i\left(\dfrac{a_1}{2} + a_2\right)}{\left(1 + a_1 + a_2\right)\sqrt{4a_2 - a_1^2}} + \frac{\gamma_r}{1 + a_1 + a_2}, \qquad (2.23)$$

$$\varphi_{12} = -\frac{2\gamma_i\left(\dfrac{a_1}{2} + a_2\right)}{\left(1+a_1+a_2\right)\sqrt{4a_2-a_1^2}} + \frac{\gamma_r}{1+a_1+a_2} \qquad (2.24)$$

and

$$\varphi_{21} = \varphi_{22} = \frac{\alpha_2}{1+a_1+a_2} . \qquad (2.25)$$

The coefficients of the other form of $H(s)$ are:

$$K = \frac{b_1 + b_2}{1+a_1+a_2} ; \quad T = \frac{1}{\sqrt{\alpha_2}} = \frac{1}{\sqrt{\gamma_r^2 + \gamma_i^2}} \qquad (2.26)$$

and

$$\xi = \frac{\alpha_1}{2\sqrt{\alpha_2}} = \frac{\gamma_r}{\sqrt{\gamma_r^2 + \gamma_i^2}} ; \qquad T_1 = \frac{\beta_1}{\beta_2} =$$

$$= \frac{\gamma_r \sqrt{4a_2-a_1^2} + \gamma_i \, (2a_2+a_1) \, (b_1-b_2)}{\left(\gamma_r^2 + \gamma_i^2\right) \sqrt{4a_2 - a_1^2}} . \qquad (2.27)$$

(In case of real roots, the function arch has to be used instead of arccos in the relations (2.20) and (2.21)!)

The examples solved heretofore on the basis of Eq. (2.7) giving the equivalent continuous system of the discrete transfer functions obtained by discrete identification ensure the system response coinciding at the sampling instants for the given input signal (here step response). The approach discussed up to now pointed at the equivalence with respect to the exciting signal and referred only indirectly to the character of the reconstructor occurring necessarily with the transformation. The transformation approach investigated

$$H(s) = \frac{\beta}{s+\alpha} = \frac{K}{1+sT}$$

Table 2-1.

		$u(\tau)$	$G(z^{-1})$	
I Impulse		$hu_k\,\delta(\tau-kh)$	$\frac{h\beta}{1-e^{-\alpha h}z^{-1}}$	$\frac{Kh/T}{1-e^{-h/T}z^{-1}}$
II Unit step	1	$mu_k+(1-m)u_{k-1}$	$\frac{\beta}{\alpha}(1-e^{-\alpha h})\frac{m+(1-m)z^{-1}}{1-e^{-\alpha h}z^{-1}}$	$K(1-e^{-h/T})\frac{m+(1-m)z^{-1}}{1-e^{-h/T}z^{-1}}$
	2	u_k	$\frac{\beta}{\alpha}(1-e^{-\alpha h})\frac{1}{1-e^{-\alpha h}z^{-1}}$	$K(1-e^{-h/T})\frac{1}{1-e^{-h/T}z^{-1}}$
	3	u_{k-1}	$\frac{\beta}{\alpha}(1-e^{-\alpha h})\frac{z^{-1}}{1-e^{-\alpha h}z^{-1}}$	$K(1-e^{-h/T})\frac{z^{-1}}{1-e^{-h/T}z^{-1}}$
	4	$0.5(u_k+u_{k-1})$	$\frac{\beta}{2\alpha}(1-e^{-\alpha h})\frac{1+z^{-1}}{1-e^{-\alpha h}z^{-1}}$	$\frac{K}{2}(1-e^{-h/T})\frac{1+z^{-1}}{1-e^{-h/T}z^{-1}}$
III Linear	1	$u_{k-1}+(u_{k-1}-u_{k-2})\frac{\tau-h(k-1)}{h}$	$\frac{\beta}{\alpha^2}\;\frac{(2h\alpha-1+e^{-\alpha h}-h\alpha e^{-\alpha h})+(1-h\alpha-e^{-\alpha h})z^{-1}}{1-e^{-\alpha h}z^{-1}}$	$K\;\frac{(2h-T+Te^{-h/T}-he^{-h/T})+(T-h-Te^{-h/T})z^{-1}}{1-e^{-h/T}z^{-1}}$
	2	$u_{k-1}+(u_k-u_{k-1})\frac{\tau-h(k-1)}{h}$	$\frac{\beta}{\alpha^2}\;\frac{(h\alpha-1+e^{-\alpha h})+(1-e^{-\alpha h}-h\alpha e^{-\alpha h})z^{-1}}{1-e^{-\alpha h}z^{-1}}$	$K\;\frac{(h-T+Te^{-h/T})+(T-Te^{-h/T}-he^{-h/T})z^{-1}}{1-e^{-h/T}z^{-1}}$

by HAYKIN and based on the basic integro-difference equation [48] shows the character of the applied reconstructor exactly.

Let the transfer function of the continuous system be

$$H(s) = \frac{\beta}{s + \alpha} , \tag{2.28}$$

then we obtain the output signal by the convolution integral, i.e.:

$$y(t) = \beta e^{-\alpha t} \int_0^t e^{\alpha\tau} u(\tau)\, d\tau . \tag{2.29}$$

By writing the equation for the time instants t=k.h and t=(k-1)h we simply get from the two equations that

$$y_k = e^{-\alpha h} y_{k-1} + \beta e^{-\alpha kh} \int_{(k-1)h}^{kh} e^{\alpha\tau} u(\tau) d\tau . \tag{2.30}$$

This equation is the basic integro-difference equation describing the discrete model exactly. It can be seen that the discrete difference equation obtained as a result depends on the evaluation of the integral in the second term. The computation of the integral, on the other hand, depends on the approximation applied to the input signal $u(\tau)$ in the interval $(k-1)\ h \leq t \leq kh$ or on the reconstructor defined by it. (In fact, a discrete model with varying parameters would correspond to the continuous system, by a given approximation of the input signal, the model is considered constant for the "average" behaviour of the system.)

Table 2-1 summarizes the various approximation possibilities of the input signal and the discrete transfer functions obtained this way. Case I means the pulse approximation without reconstructor. With this transformation involving the worst approximation, even the transformation of the gain factor is biased (zero frequency). Case II indicates the step approximations, when the input signal is approximated by a constant in the sampling interval. m shows how the values taken on the two end points of the interval are considered. The table presents the special cases

$$m = 1,\ 0,\ 0.5\ .$$

The case $m=0$ means a reconstructor corresponding to a zero order holding element and corresponds to a step response equivalent transformation according to (2.11). Case III implies linear approximation, viz. means first order holding element. Version 1 refers to the extrapolation, version 2 to the interpolation. This latter ensures a ramp response equivalent transformation. By comparing the discrete transfer functions in the table with the forms according to (2.8) or (2.14), the rules of computation of the coefficients are simply obtained. Calculations can relatively simply be performed even for second order systems, of course.

On the basis of the above, we can state that the denominator of the discrete transfer function can, by using the above-described methods, be transformed into the continuous system in every case in the same way (by exponential transformation pole by pole) and only the transformation of the coefficients of the numerator depends on the reconstructor applied or the input signal giving the equivalence. As a discrete transfer function can be expanded into partional fractions (subsystems), the above mentioned relations enable the equivalent continuous subsystems to be defined and the complete transfer function of the continuous system to be built from them.

(Note though that α can be considered complex pole for the first-order systems and then the formulas remain valid. It is, however, more adviseable to apply the transformation using the resolution into first and second order subsystems which requires real arithmetics. We do not intend to discuss now the case of multiple poles.)

The transformation of the poles comes therefore about according to the relation

$$s = \frac{1}{h} \ln z . \tag{2.31}$$

By expanding it and considering it up to the first-order term, we obtain the approximating relation

$$s \cong \frac{2}{h} \; \frac{1 - z^{-1}}{1 + z^{-1}} \tag{2.32}$$

of the bilinear z-transformation [64]. By means of this, carrying out in G(z) the substitution

$$z^{-1} = \frac{2 - hs}{2 + hs} \tag{2.33}$$

the equivalent continuous system H(s) can be simply obtained. (In the case of the ideal integrator $H(s) = \frac{1}{s}$ the transformation of the average value II/4 in Table 2-1 is exactly the same as the bilinear z-transformation). 1/s now passes over to the integration according to the trapezoid rule.

By applying (2.33) to the first-order discrete model (2.8), we obtain H(s) according to (2.15), where

$$K = \frac{b_1}{1 + a_1}; \quad T = \frac{1-a_1}{1 + a_1}\frac{h}{2} \quad \text{and} \quad T_1 = -\frac{h}{2} . \tag{2.34}$$

The second-order system (2.18) can be transformed to the continuous form

$$F(s) = K\,\frac{1 + 2\zeta T_1 s + T_1^2\, s^2}{1 + 2\xi T s + T^2\, s^2} \tag{2.35}$$

with the substitution (2.33), where

$$K = \frac{b_1 + b_2}{1 + a_1 + a_2}; \qquad T = \frac{h}{2}\sqrt{\frac{1 - a_1 + a_2}{1 + a_1 + a_2}} \quad \text{and}$$

$$\xi = \frac{1 - a_2}{\sqrt{(1 + a_2)^2 - a_1^2}} \tag{2.36}$$

or

$$T_1 = \frac{h}{2}\sqrt{\frac{b_2 - b_1}{b_1 + b_2}} \qquad \text{and} \qquad \zeta = -\frac{b_2}{\sqrt{b_2^2 - b_1^2}} . \tag{2.37}$$

The substitution according to (2.33) is simple to be algorithmized in a computer and can be applied directly, there is no need to decompose the transfer function to subsystems. The quantity of the calculations is considerably less than with the transformation methods previously discussed - where the decomposition to subsystems is unavoidable so that the determination of the poles is needed - therefore, its use for on-line applications has obvious advantages. Moreover, SINHA has also demonstrated with simulation tests [94] that this transformation principle yielded for various types of excitations more exact results than the other methods.

The bilinear transformation distorts the frequency scale [64], [65]. This effect can be eliminated by determining the roots of the numerator and denominator of the continuous system obtained by the substitution (2.33), thereupon by rescaling the imaginary part μ with the complex roots according to the relation

$$\omega = \frac{2}{h} \, tg \, \left(\frac{h}{2} \mu \right) \tag{2.38}$$

and by multiplying the new roots with each other [65].

All the transformation methods discussed heretofore, used a deterministic way of approach. HSIA took advantage also of the spectral properties of the input signal [50] although with his method, it must be assumed of the input signal that it has a much vider bandwith than the system. In this approach, the equivalence of the power density spectrum of the output signals of the continuous system and discrete model is prescribed on the sampling frequency (and its multiples).

Let the power density spectrum of the signal $y(t)$ be $\Phi_{yy}(s)$, that of the sampled y_k output signal $\Psi_{yy}(z)$. On the basis of the relation (2.3), we prescribe the fulfilment of the equivalence

$$\Psi_{yy}(z) = S\{\Phi_{yy}(s)\} \tag{2.39}$$

i.e. the equality

$$G(z^{-1})\ G(z)\ \Psi_{uu}(z) = S\{\Phi_{uu}(s)\ H(s)\ H(-s)\} \qquad (2.40)$$

Here $\Phi_{uu}(s)$ and $\Psi_{uu}(z)$ are the power density spectra of the continuous input signal $u(t)$ and the sampled signal u_k. (It is practical to perform the transformation indicated by (2.40) partitioning $\Phi_{yy}(s)$ into two parts [50].)

For the case of $\Phi_{uu}(s) = 1$, i.e. for the white noise input signal (practically when the input signal has a much vider bandwith, than the system), HSIA has defined the conversion relations of the (2.8) first-order discrete model to the continuous system (2.9) , according to which

$$\alpha_1 = -\frac{\ln(-a_1)}{h}\ ; \qquad \beta_1 = \frac{b_1}{h}\sqrt{\frac{-2\ln(-a_1)}{1 - a_1^2}} \qquad (2.41)$$

or

$$K = \frac{b_1}{1 + a_1} \cdot \sqrt{\frac{-2(1 + a_1)}{\ln(-a_1)(1-a_1)}}; \quad T = -\frac{h}{\ln(-a_1)} . \qquad (2.42)$$

It is well seen that the gain factor is biased comparing to the step response equivalent transformation. Its reason is that $\Phi_{uu}(s)$ is approximated. The exact knowledge of the spectrum of the input signal substantially increases the importance of this method.

All approaches until now have discussed the transformation by subsystems and the investigation of multiple poles could not be arrived at because of its complexity. JEŽEK [54] pointed out that the integro-difference equation describing the sampling system can be obtained by direct integration of state equations and consequently also the relations of equivalent transformations. Following this way the state space algorithms, easily utilizable in computational techniques, of the step and ramp response equivalent transformation are also given.

Consider any state space description of a single input-single output continuous system in the form

$$\dot{\underline{x}}(t) = \underline{\underline{A}}\,\underline{x}(t) + \underline{b}\,u(t) \tag{2.43}$$

$$y(t) = \underline{c}^T\underline{x}(t) + \beta_o\,u(t)\,. \tag{2.44}$$

(Here T denotes the transposition.) The solution of the continuous state equations for the sampling interval $kh \leqq t \leqq (k+1)h$

$$\underline{x}((k+1)h) = e^{\underline{\underline{A}}h}\,\underline{x}(kh) + \int_{k.h}^{(k+1)h} e^{\underline{\underline{A}}((k+1)h-\tau)}\,\underline{b}\,u(\tau)\,d\tau \tag{2.45}$$

i.e. the integro-vector difference equation of the discrete system

$$\underline{x}_{k+1} = e^{\underline{\underline{A}}h}\underline{x}_k + \int_{k.h}^{(k+1)h} e^{\underline{\underline{A}}((k+1)h-\tau)}\,u(\tau)\,d\tau\,.\underline{b} =$$

$$= e^{\underline{\underline{A}}h}\,\underline{x}_k + \underline{\underline{Q}}\left[u(\tau)\,,\ \underline{\underline{A}},\ h\right]\underline{b}. \tag{2.46}$$

By comparing this latter result with the state equations of a discrete-time linear system

$$\underline{x}_{k+1} = \underline{\underline{F}}\, \underline{x}_k + \underline{g}\, u_k \tag{2.47}$$

$$y_k = \underline{c}^T \underline{x}_k + b_o\, u_k \tag{2.48}$$

and evaluating the so-called input integral

$$\underline{\underline{G}}\left[u(\tau),\ \underline{\underline{A}},\ h \right] = \int_{kh}^{(k+1)h} e^{\underline{\underline{A}}((k+1)h-\tau)}\, u(\tau)d\tau =$$

$$= \int_0^h e^{\underline{\underline{A}}(h-\vartheta)}\, u(kh+\vartheta)d\vartheta \tag{2.49}$$

for a given type approximation of the input signal $u(t)$ in the interval $kh \leq t \leq (k+1)h$ unambiguous relations can be obtained between the continuous and discrete state equations by the comparison of coefficient matrices, assuming even the time-independence of the applied approximation.

If suitable canonical equations are applied for comparison, then the use of common notations in the output equations (2.44) and (2.45) is justified. (Then $\underline{c}$ usually does not contain system parameter, further the coefficients of $u(t)$ and u_k coincide.) Otherwise, the conversion to such a form can be carried out by a simple transformation [24].

Comparing the Eqs. (2.44) and (2.45) the transformation rule of $\underline{\underline{F}}$ is obtained in the form

$$\underline{\underline{A}} = \frac{1}{h} \ln(\underline{\underline{F}}) \tag{2.50}$$

which is independent from the approximation of the input signal, according to the experience with the forms decomposed into subsystems. Computer subroutines ready to compute the matrix function $\ln(\underline{\underline{F}})$, which can be defined, among others, by its series, are available. It is a necessary condition of the existence of $\underline{\underline{A}}$ that all eigenvalues of $\underline{\underline{F}}$ be located within the unit circle. (In case of negative real root, $\underline{\underline{A}}$ cannot be computed.)

Investigate now the input integral in case of two kinds of approximation of $u(t)$. First let $u(t)$ be constant in the complete sampling period, i.e.

$$u(kh + \vartheta) \cong u(kh) = u_k$$

which assumption is needed for the step response equivalent transformation.

Then

$$\underline{Q}\left[u(\tau),\ \underline{\underline{A}},\ h\right] = \int_0^h e^{\underline{\underline{A}}(h-\vartheta)} u_k \, d\vartheta = u_k \int_0^h e^{\underline{\underline{A}}(h-\vartheta)} \, d\vartheta =$$

$$= u_k \left[-\underline{\underline{A}}^{-1} e^{\underline{\underline{A}}(h-\vartheta)}\right]_0^h = \underline{\underline{A}}^{-1} (e^{\underline{\underline{A}}h} - \underline{\underline{I}})\, u_k, \tag{2.51}$$

where $e^{\underline{\underline{0}}} = \underline{\underline{I}}$ was taken into account [24]. Eq. (2.44) takes now the form

$$\underline{x}_{k+1} = e^{\underline{\underline{A}}h} \underline{x}_k + \underline{\underline{A}}^{-1} (e^{\underline{\underline{A}}h} - \underline{\underline{I}})\, \underline{b}\, u_k \tag{2.52}$$

and by comparing this with (2.47), we obtain

$$\underline{b} = \frac{1}{h} \ln(\underline{\underline{F}})\ (\underline{\underline{F}} - \underline{\underline{I}})^{-1}\ \underline{g}. \tag{2.53}$$

Furthermore now $\beta_o = b_o$.

Approximate now the input signal in the sampling interval according to the linear interpolation in the form

$$u(kh + \vartheta) \cong u_k + \frac{u_{k+1} - u_k}{h} \tag{2.54}$$

which corresponds to the ramp response equivalent transformation. The input integral is now

$$\underline{\underline{Q}}\left[u(\tau),\ \underline{\underline{A}},\ h\right] \cong \int_0^h e^{A(h-\vartheta)}\left[u_k + \frac{u_{k+1}-u_k}{h}\vartheta\right] d\vartheta =$$

$$= u_k \int_0^h e^{\underline{\underline{A}}(h-\vartheta)} d\vartheta + \frac{u_{k+1}-u_k}{h} \int_0^h e^{\underline{\underline{A}}(h-\vartheta)} \vartheta d\vartheta . \tag{2.55}$$

After not very complex computations we obtain that

$$\underline{\underline{Q}}\left[u(\tau),\ \underline{\underline{A}},\ h\right] \cong \{ \underline{\underline{A}}^{-1}\ (e^{\underline{\underline{A}}h} - \underline{\underline{I}}) - \frac{1}{h}\left[\underline{\underline{A}}^{-2}(e^{\underline{\underline{A}}h} - \underline{\underline{I}}) - h\underline{\underline{A}}^{-1}\right]\} u_k +$$

$$+ \frac{1}{h}\left[\underline{\underline{A}}^{-2}\ (e^{\underline{\underline{A}}h} - \underline{\underline{I}}) - h\underline{\underline{A}}^{-1}\right] u_{k+1} = \underline{\underline{Q}}_1 u_k + \underline{\underline{Q}}_2 u_{k+1}. \tag{2.56}$$

Thus for (2.47) the integro-difference equation has [illegible] a state equation

$$\underline{x}_{k+1} = \underline{\underline{F}}\, \underline{x}_k + \underline{\underline{Q}}_1\, \underline{b}\, u_k + \underline{\underline{Q}}_2 \underline{b}\, u_{k+1} \,. \qquad (2.57)$$

Introducing the state vector

$$\underline{\tilde{x}}_{k+1} = \underline{x}_{k+1} - \underline{\underline{Q}}_2\, \underline{b}\, u_{k+1} \qquad (2.58)$$

the Eqs. (2.47) and (2.48) change in the following way:

$$\underline{\tilde{x}}_{k+1} = \underline{\underline{F}}\, \underline{\tilde{x}}_k + (\underline{\underline{F}}\, \underline{\underline{Q}}_2 + \underline{\underline{Q}}_1)\, \underline{b}\, u_k \qquad (2.59)$$

$$y_k = \underline{c}^T\, \underline{\tilde{x}}_k + (\underline{c}^T\, \underline{\underline{Q}}_2\, \underline{b} + \beta_o)\, u_k \,. \qquad (2.60)$$

The transformation relation (2.50) remains henceforward valid but instead of (2.53) the quantities

$$\underline{b} = (\underline{\underline{F}}\, \underline{\underline{Q}}_2 + \underline{\underline{Q}}_1)^{-1}\, \underline{g} = \frac{1}{h}\left[\ln(\underline{\underline{F}})\right]^2 (\underline{\underline{F}}-\underline{\underline{I}})^{-2}\, \underline{g} \qquad (2.61)$$

and

$$\beta_o = b_o - \underline{c}^T\, \underline{\underline{Q}}_2\, \underline{b} = b_o + \underline{c}^T\, (\underline{\underline{F}}-\underline{\underline{I}})^{-1} \left[\ln(\underline{\underline{F}})(\underline{\underline{F}}-\underline{\underline{I}})^{-1} - \underline{\underline{I}}\right] \underline{g} \qquad (2.62)$$

have to be used.

Both above presented state space transformations transform - as we have seen with the subsystems - the gain factor of the system without error which can easily be checked on the basis of the equation

$$K = \beta_o - \underline{c}^T \underline{\underline{A}}^{-1} \underline{b} = b_o - \underline{c}^T (\underline{\underline{F}}-\underline{\underline{I}})^{-1} \underline{g} \ . \qquad (2.63)$$

On the basis of the above described state space transformation methods, a discrete transfer function

$$G(z) = \frac{B(z)}{A(z)} = \frac{b_o + b_1 z^{-1} + \dots + b_n z^{-n}}{1 + a_1 z^{-1} + \dots + a_n z^{-n}} \qquad (2.64)$$

can be transformed into the equivalent continuous form

$$H(s) = \frac{\beta(s)}{\alpha(s)} = \frac{\beta_o s^n + \beta_1 s^{n-1} + \dots + \beta_n}{s^n + \alpha_1 s^{n-1} + \dots + \alpha_n} \qquad (2.65)$$

according to the following algorithm:

1. On the basis of the coefficients of (2.64) we construct the coefficient matrix of the discrete state Eqs. (2.45), (2.46), e.g. in the forms

$$\underline{\underline{F}} = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & . & . & . & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & . & . & . & -a_1 \end{bmatrix} \qquad (2.66)$$

$$\underline{g} = [0, 0, \ldots, 0, 1]^T \tag{2.67}$$

and

$$\underline{c} = [b_n - b_o a_n, \ldots, b_1 - b_o a_1]^T. \tag{2.68}$$

2. We change over to the continuous state equations on the basis of the Eqs.

$$\underline{\underline{A}} = \frac{1}{h} \ln(\underline{\underline{F}}) \tag{2.69}$$

and

$$\underline{b} = \begin{cases} \frac{1}{h} \ln(\underline{\underline{F}}) (\underline{\underline{F}} - \underline{\underline{I}})^{-1} \underline{g} & \text{(step response equivalent)} \\ \frac{1}{h} [\ln(\underline{\underline{F}})]^2 (\underline{\underline{F}} - \underline{\underline{I}})^{-2} \underline{g} & \text{(ramp response equivalent)} \end{cases} \tag{2.70}$$

and

$$\beta_o = \begin{cases} b_o & \text{(step response equivalent)} \\ b_o - \underline{c}^T (\underline{\underline{F}} - \underline{\underline{I}})^{-1} \underline{g} & \text{(ramp response equivalent)} \end{cases} \tag{2.71}$$

3. Then the continuous system obtained this way and given by $\underline{\underline{A}}$ and $\underline{b}$ is transformed into the canonical form of phase-variables $\underline{\underline{A}}^*$ and $\underline{b}^*$ by certain standard procedure [23], [24], e.g. with the transformation matrix

$$\underline{\underline{T}} = \begin{bmatrix} \underline{c}^T \\ \underline{c}^T \underline{\underline{A}} \\ \vdots \\ \underline{c}^T \underline{\underline{A}}^{n-1} \end{bmatrix} . \tag{2.72}$$

In the canonical form

$$\underline{\underline{A}}^* = \left[\begin{array}{c|c} \underline{0} & \underline{\underline{I}} \\ \hline \multicolumn{2}{c}{-\underline{k}^T} \end{array}\right] \tag{2.73}$$

where

$$\underline{k} = [\alpha_n, \alpha_{n-1}, \ldots, \alpha_1]^T \tag{2.74}$$

is the so-called canonical vector. $\underline{k}$ vector is obtained in the following way:

$$\underline{k}^T = -[\underline{c}^T \underline{\underline{A}}^n] \underline{\underline{T}}^{-1} \tag{2.75}$$

further

$$\underline{b}^* = \underline{\underline{T}} \cdot \underline{b} . \tag{2.76}$$

We have already got the denominator of H(s), its numerator is given by the following relation:

$$\underline{g} = [\gamma_1, \gamma_2, \ldots, \gamma_n]^T = \underline{\underline{P}}\ \underline{b}^* = \underline{\underline{P}}\ \underline{\underline{T}}\ \underline{b} \tag{2.77}$$

where

$$\underline{\underline{P}} = \begin{bmatrix} 1 & 0 & . & . & . & 0 \\ \alpha_1 & 1 & . & . & . & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-1} & \alpha_{n-2} & . & . & . & 1 \end{bmatrix} \tag{2.78}$$

Finally the coefficients of the numerator of $H(s)$ can be calculated according to

$$\beta_i = \gamma_i + \beta_o\ \alpha_i; \qquad i = 1, 2, \ldots, n \tag{2.79}$$

where β_o is according to (2.71).

The course of the algorithm is summarized in Fig. 2-3.

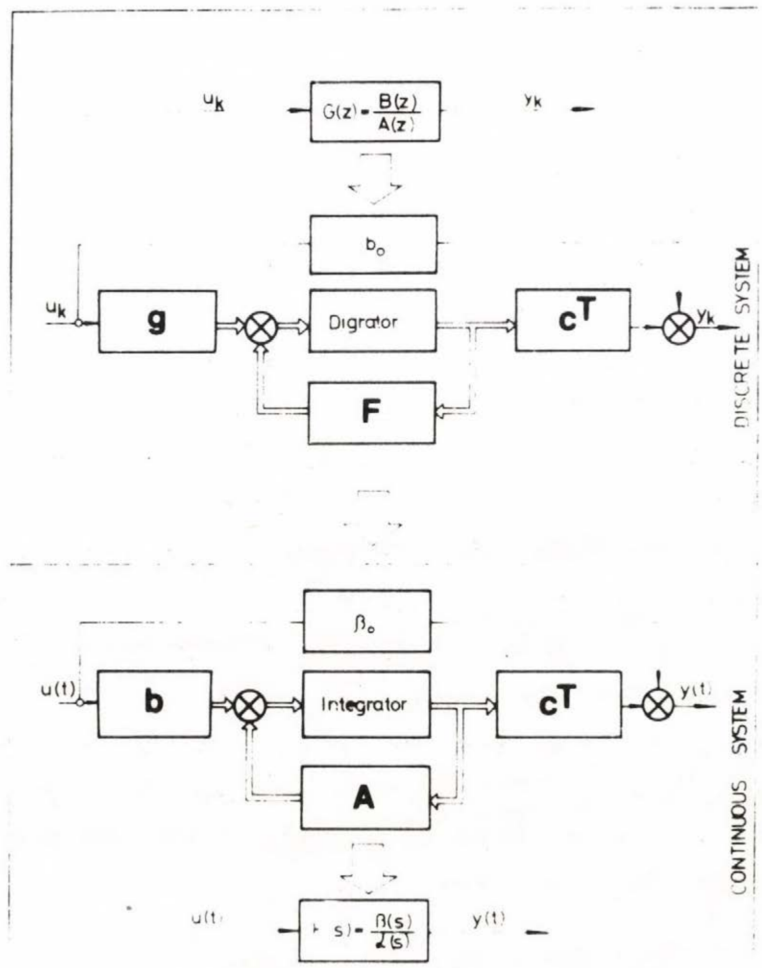
u_k
$G(z) = \frac{B(z)}{A(z)}$
y_k
b_o
u_k
g
Digrator
c^T
y_k
F
DISCRETE SYSTEM
β_o
u(t)
b
Integrator
c^T
y(t)
A
CONTINUOUS SYSTEM
u(t)
y(t)

Fig. 2-3.

III. OPTIMIZATION OF SAMPLING TIME WITH RESPECT TO THE SENSITIVITY OF z-s TRANSFORMATION

In the preceding chapter we have seen that except the bilinear transformation the poles of the discrete transfer function can be transformed to the continuous system with an exponential transformation. Only the transformation of the numerator depended on the applied reconstructor or input signal producing the equivalence. In this chapter the transformation sensitivity of the poles, zeros of the discrete transfer function will be investigated as a function of the sampling period. As the sampling period is optimized from identification purpose, the sensitivity function of the transformation will be minimized as a criterion at the real parameter values of the process as a function of h. The discrete transfer function namely is obtained as a result of the estimation so that the aim is to choose a sampling period at which the uncertainty of the discrete poles and zeros should occur as less as possible in the continuous system obtained by transformation.

3.1 Minimization of the sensitivity of pole transformation

Our investigations deal first with the effect of the sampling time on the transformation of the discrete transfer function to a step response equivalent continuous system.

By summarizing once more the results of relations (2.8) - (2.13) consider first a first-order system:

$$G(z) = \frac{b_1 z^{-1}}{1+a_1 z^{-1}} \longrightarrow H(s) = \frac{\beta_1}{\alpha_1+s} = \frac{\beta_1/\alpha_1}{1+1/\alpha_1\, s} = K\,\frac{1}{1+sT}$$

$$\alpha_1 = -\frac{\ln(-a_1)}{h}\;;\qquad \beta_1 = -\frac{b_1\ln(-a_1)}{h(1+a_1)} = \frac{b_1\,\alpha_1}{1+a_1}\;;$$

$$K = \frac{\beta_1}{\alpha_1} = \frac{b_1}{1+a_1} = G(1) = H(0);$$

$$T = \frac{1}{\alpha_1} = -\frac{h}{\ln(-a_1)}\,. \tag{3.1-1}$$

Hereinafter constraints are used for the location of the poles from the viewpoints of realizability and stability. In continuous case unstable systems are not dealt with (only with roots falling to the left half-plane), and from the discrete transfer functions those are rejected whose poles are located outside the unit circle. We must say some words here about the poles falling to the left side of the unit circle. Several authors have already reported [13] that the discrete transfer functions with a stable root on the left side usually refer to the overestimated order. These experiences have been confirmed also by the results of several coworkers of the Technical University of Budapest. In case of real roots - as we shall see - the results show unanimously that the roots on the left side should not be considered. This phenomenon is easily understandable since at the exponential transformation of the poles the negative real axis in the z plane corresponds to the SHANNON' boundary frequency [22], i.e. to the marginal line of the principal band in the s plane which does not belong any more to the transformation.

For complex roots the situation is not so unambiguous. For deterministic case the decrease of the SHANNON' boundary cannot be admitted, only with systems having stochastic disturbances. Results can be obtained, according to which, in consequence of the decrease of the information, certain areas of the left half-circle have to be given up (cf. Chapter 4), while the definition of these separating forms is extremely difficult.

Because of the above, there is a restriction for the pole z_1 of the first-order $G(z)$ and thus also for the parameter a_1:

$$0 < z_1 = -a_1 < 1 , \qquad (3.1\text{-}2)$$

i.e. the pole z_1 can be located only on the right side of the real axis within the unit circle. Accordingly

$$-1 < a_1 < 0 . \qquad (3.1\text{-}3)$$

Let us express now on the basis of (3.1-1) the pole of the discrete system:

$$z_1 = -a_1 = e^{-\frac{h}{T}} = e^{-x} \qquad (3.1\text{-}4)$$

where the relative sampling rate

$$x = \frac{h}{T} \tag{3.1-5}$$

have been introduced.

Fig. 3.1-1 shows the character of the function relation (3.1-4).

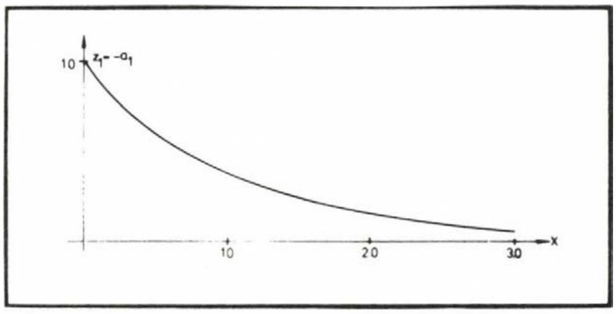

Fig. 3.1-1

If the function $z_1(x)$ in Fig. 3.1-1 is represented together with the unit circle the location of the pole of the discrete system can be obtained directly. Therefore, the location of the pole of the first-order system depends only on the relative sampling rate and is independent from the parameters of the continuous system. (Cf. Fig. 3.1-2). It is obvious that unstable system can not be obtained by any sampling time. Besides, the pole z_1 can be plotted easily from the figure.

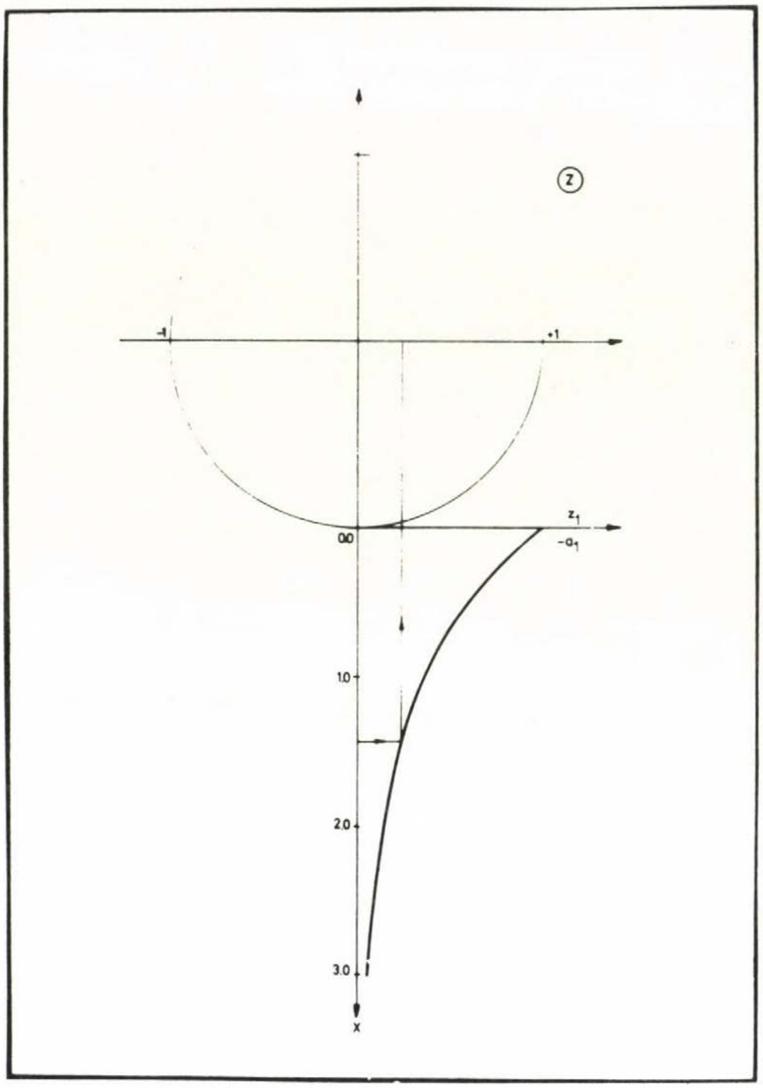

Fig. 3.1-2

According to the transformation of minimal sensitivity mentioned in the introduction, it would be good if the uncertainty (because of estimation errors) arising in the root z_1 would occur as slightly as possible in the corresponding root, s_1 of the continuous system. As the information - because of the identification - starts from the continuous system, whereafter the continuous form is obtained through the discrete model - by reason of the estimation techniques - the transformation of the uncertainty takes the form shown in Fig. 3.1-3.

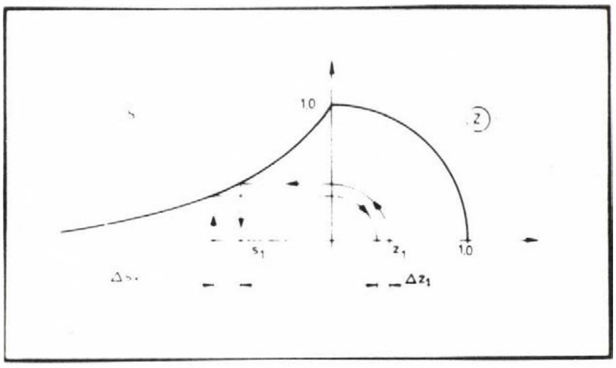

Fig. 3.1-3

The quantitative relation between the roots on the basis of the relations (3.1-1) is

$$s_1 = -\alpha_1 = -\frac{1}{T} = \frac{\ln(-a_1)}{h} = \frac{\ln(z_1)}{h}, \qquad (3.1\text{-}6)$$

where s_1 is the pole of the continuous system.

The desired optimality is considered through the sensitivity $\partial s_1/\partial z_1$ in defining the relative sensitivity function

$$E \triangleq \frac{\left|\frac{\partial s_1}{\partial z_1}\right|}{|s_1|} = \frac{\left|\frac{\partial s_1}{\partial z_1}\right|}{\frac{1}{T}} . \qquad (3.1\text{-}7)$$

Here, on the one hand, the absolute value of the partial derivative was taken into account, on the other hand it was related to the cutting frequency 1/T (i.e. to the continuous pole itself) in order to obtain a dimensionless quantity, the relative error. Calculate the function E

$$E = E(x) = T \left| \frac{\partial s_1}{\partial z_1} \right| = T \left| \frac{1}{h} \cdot \frac{1}{z_1} \right| = T \frac{1}{h|z_1|} =$$

$$= \frac{T\, e^{h\alpha_1}}{h} = \frac{e^{h/T}}{h/T} = \frac{e^x}{x} \qquad (3.1\text{-}8)$$

which obviously is only the function of the relative sampling rate. $E(x)$ has to be minimized for the optimization of the sampling rate:

$$E(x) \longrightarrow \min_x \, . \qquad (3.1\text{-}9)$$

Let us form the derivative by x:

$$\frac{dE(x)}{dx} = \frac{d}{dx}\left(x^{-1}\, e^x\right) = -\frac{1}{x^2}\, e^x + \frac{1}{x}\, e^x = 0 \qquad (3.1\text{-}10)$$

and the optimum from the above condition is

$$x_{opt} = 1 \qquad (3.1\text{-}11)$$

which means that the optimal sampling rate is

$$h_{opt} = T \, . \qquad (3.1\text{-}12)$$

Alike the sensitivity function we can use also its reciprocal:

$$Q(x) = \frac{1}{E(x)} = \frac{\frac{1}{T}}{\left|\frac{\partial s_1}{\partial z_1}\right|} = \frac{1}{T\left|\frac{\partial s_1}{\partial z_1}\right|} \longrightarrow \max_{x} \qquad (3.1\text{-}13)$$

which is called insensitivity function and which, of course, has to be maximized by x. On computational considerations we are going to use this hereinafter. Fig. 3.1-4 shows the two functions for the first order system investigated.

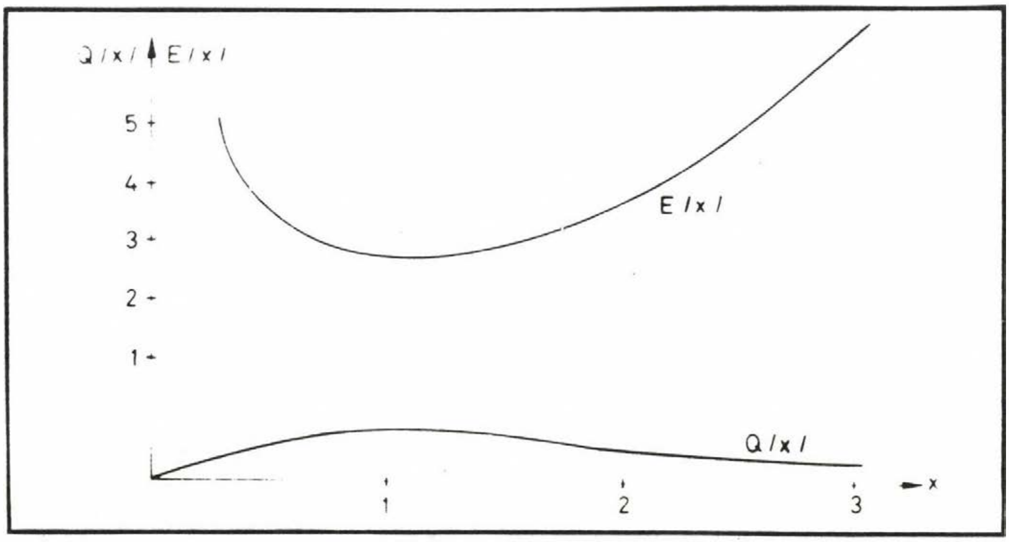

Fig. 3.1-4

Consider now Fig. 3.1-2 with the curve of function $Q(x)$. (Cf. Fig. 3.1-5!). As a result of optimization, we have obtained that x = 1 or h = T is the optimal sampling time. Accordingly from Eq. (3.1-4)

$$z_{1\ opt} = e^{-x}\Big|_{x_{opt}} = e^{-1} = \frac{1}{e} = 0{,}36788. \qquad (3.1\text{-}14)$$

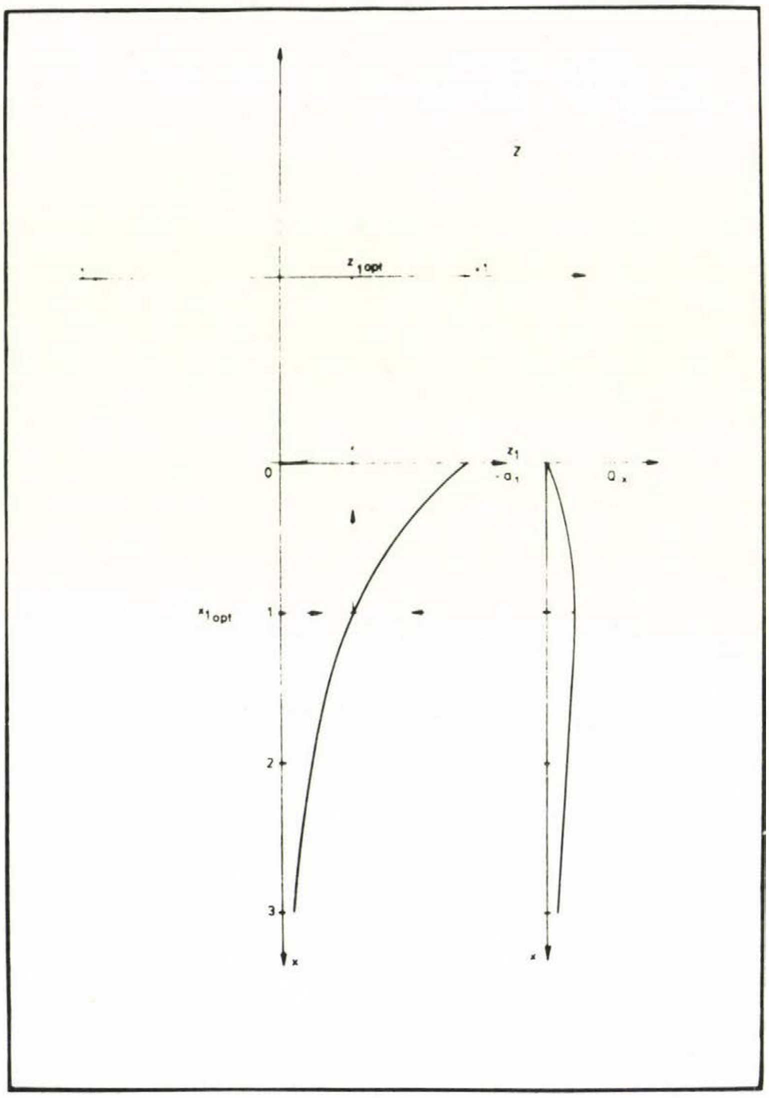

Fig. 3.1-5

In case of several disjunct real poles

$$E(x) = \frac{\prod_{i=1}^{n} \left| \frac{\partial s_i}{\partial z_i} \right|}{\frac{1}{T_o}} = T_o \prod_{i=1}^{n} \left| \frac{\partial s_i}{\partial z_i} \right| \longrightarrow \min_{x} \qquad (3.1\text{-}15)$$

can be used as a sensitivity function, whence

$$Q(x) = \frac{1}{E(x)} = \frac{\frac{1}{T_o}}{\prod_{i=1}^{n} \left| \frac{\partial s_i}{\partial z_i} \right|} = \frac{1}{T_o \prod_{i=1}^{n} \left| \frac{\partial s_i}{\partial z_i} \right|} \xrightarrow[x]{} \max. \tag{3.1-16}$$

Here s_i, z_i refers to the i-th subsystem (pole), n is the number of poles, further $1/T_o$ is a given point of the frequency scale, i.e. T_o is a function of T_i-s in order to form the relative reference basis.

Thus, in case of several real poles, the resultant sensitivity function was considered as the product of the sensitivities of the particular subsystems. (Note that this form is in accordance with the logarithmic sensitivity usual in the sensitivity analysis which is additive).

In the calculation of $Q(x)$

$$\prod_{i=1}^{n} \left| \frac{\partial s_i}{\partial z_i} \right| = \prod_{i=1}^{n} \left| \frac{1}{h} \quad \frac{1}{z_i} \right| = \frac{1}{h^n} \prod_{i=1}^{n} e^{h/T_i} . \tag{3.1-17}$$

It is easy to see that in Eqs. (3.1-15) and (3.1-16) a reference basis T_o depending on T_i-s has to be chosen. On basis of (3.1-17)

$$E(x) = T_o \prod_{i=1}^{n} \left| \frac{\partial s_i}{\partial z_i} \right| = T_o \prod_{i=1}^{n} \frac{1}{h} \; e^{h/T_i} =$$

$$= \frac{T_o}{h^n} \prod_{i=1}^{n} e^{h/T_i} \cdot = E(h) \tag{3.1-18}$$

which developed further

$$E(x) = \frac{T_o}{h^n} \; e^{h\left(\sum\limits_{i=1}^{n} \frac{1}{T_i}\right)} = E(h). \tag{3.1-19}$$

Let us seek the minimum of

$$\frac{d\,E(x)}{dx} = \frac{d}{dx} \; \frac{T_o}{h^n} \; e^{h\left(\sum\limits_{i=1}^{n} \frac{1}{T_i}\right)} = 0. \tag{3.1-20}$$

As the meaning of x is now undefined, let us seek first the optimum in the function of h

$$\frac{\partial E(h)}{\partial h} = \frac{\partial}{\partial h} \; \frac{T_o}{h^n} \; e^{h\left(\sum\limits_{i=1}^{n} \frac{1}{T_i}\right)} = 0 \tag{3.1-21}$$

whence

$$T_o\left[- \frac{n}{h^{(n+1)}} \; e^{h\left(\sum\limits_{i=1}^{n} \frac{1}{T_i}\right)} + \frac{1}{h^n}\left(\sum_{i=1}^{n} \frac{1}{T_i}\right) e^{h\left(\sum\limits_{i=1}^{n} \frac{1}{T_i}\right)} \right] = 0 \tag{3.1-22}$$

i.e.

$$- \frac{n}{h^{(n+1)}} + \frac{\sum\limits_{i=1}^{n} \frac{1}{T_i}}{h^n} = 0 . \tag{3.1-23}$$

From this latter equation follows:

$$h_{opt} = \frac{n}{\sum\limits_{i=1}^{n} \frac{1}{T_i}} = \frac{1}{\frac{1}{n}\sum\limits_{i=1}^{n} \frac{1}{T_i}} . \tag{3.1-24}$$

On the basis of this latter expression, it is reasonable to choose the "averaged" time constant as T_o, i.e.:

$$T_o = \frac{n}{\sum\limits_{i=1}^{n} \frac{1}{T_i}} = \frac{1}{\frac{1}{n}\sum\limits_{i=1}^{n} \omega_i} = \frac{1}{\frac{1}{n}\sum\limits_{i=1}^{n} \frac{1}{T_i}} = \frac{1}{\omega_o} . \tag{3.1-25}$$

Here ω_i is the cutting frequency calculated on the basis of the time constant T_i of the i-th pole, and ω_o is the arithmetical mean of the cutting frequencies. Here T_o assigns actually a reference basis to the medium frequency domain of the continuous system. On this basis

$$\frac{h_{opt}}{T_o} = x_{opt} = 1 \tag{3.1-26}$$

and formally the same result is obtained as for one pole. This therefore means that in case of several disjunct poles this optimal sampling time coincides with the fictive time constant corresponding to the arithmetical mean of the cutting frequencies.

Now every pole was taken into account with equal weight. It can be easily understood that by applying various w_i weights

$$h_{opt} = \frac{1}{\frac{1}{n} \sum_{i=1}^{n} \frac{w_i}{T_i}} = \frac{n}{\sum_{i=1}^{n} \frac{w_i}{T_i}} \tag{3.1-27}$$

is obtained as optimal sampling period, where w_i/T_i represents a first-order subsystem with real pole. It is practical to ensure the condition

$$\sum_{i=1}^{n} w_i = 1; \qquad w_i \geqq 0 . \tag{3.1-28}$$

(The residues belonging to the single poles can be chosen as w_i).

Following the preceding train of thoughts, consider now the step response equivalent transformation of second-order systems from the viewpoint of sensitivity.

The transformation of discrete function

$$G(z) = \frac{b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \tag{3.1-29}$$

into the continuous second-order system

$$H(s) = \frac{\beta_1 s + \beta_2}{s^2 + \alpha_1 s + \alpha_2} = K \frac{1 + sT_1}{1 + 2\xi Ts + T^2 s^2} \tag{3.1-30}$$

means the following relations between the parameters by assuming complex poles:

$$\alpha_1 = 2\gamma_r = -\frac{1}{h}\ln(a_2) \tag{3.1-31}$$

$$\alpha_2 = \gamma_i^2 + \gamma_r^2 \tag{3.1-32}$$

where

$$\gamma_r = -\frac{1}{2h}\ln(a_2) \tag{3.1-33}$$

$$\gamma_i = -\frac{1}{h}\,\mathrm{arc\,cos}\left(-\frac{a_1}{2\sqrt{a_2}}\right). \tag{3.1-34}$$

Further

$$T = \frac{1}{\sqrt{\alpha_2}} = \frac{1}{\sqrt{\gamma_r^2 + \gamma_i^2}} \tag{3.1-35}$$

and

$$\xi = \frac{\alpha_1}{2\sqrt{\alpha_2}} = \frac{\gamma_r}{\sqrt{\gamma_r^2 + \gamma_i^2}} \tag{3.1-36}$$

according to the relations (2.18) - (2.27).

(The case of real poles is not discussed here in particular for (3.1-26) includes it.)

For the better understanding of the relations and the more clear interpretation of the results, consider first the poles of (3.1-29). The roots of the equation

$$1 + a_1 z^{-1} + a_2 z^{-2} = z^2 + a_1 z + a_2 = 0 \qquad (3.1\text{-}37)$$

are

$$z_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} = -\frac{a_1}{2} \pm \sqrt{\frac{a_1^2}{4} - a_2}\ . \qquad (3.1\text{-}38)$$

In a complex case the square of the radius:

$$r^2 = \mathrm{Re}^2 + \mathrm{Im}^2 = \frac{a_1^2}{4} + a_2 - \frac{a_1^2}{4} = a_2 \qquad (3.1\text{-}39)$$

namely then

$$z_{1,2} = \mathrm{Re} \pm j\,\mathrm{Im} = -\frac{a_1}{2} \pm j\sqrt{a_2 - \frac{a_1^2}{4}}\ . \qquad (3.1\text{-}40)$$

Thus

$$\mathrm{Im} = \sqrt{a_2 - \frac{a_1^2}{4}} \qquad (3.1\text{-}41)$$

and

$$\mathrm{Re} = -\frac{a_1}{2}\ . \qquad (3.1\text{-}42)$$

The condition of the existence of the complex root is the fulfilment of

$$a_1^2 - 4a_2 < 0 \tag{3.1-43}$$

whence the domain of the complex roots results on the plane of the parameters a_1, a_2 according to

$$a_2 > \frac{1}{4} a_1^2 \, . \tag{3.1-44}$$

Fig. 3.1-6 shows this.

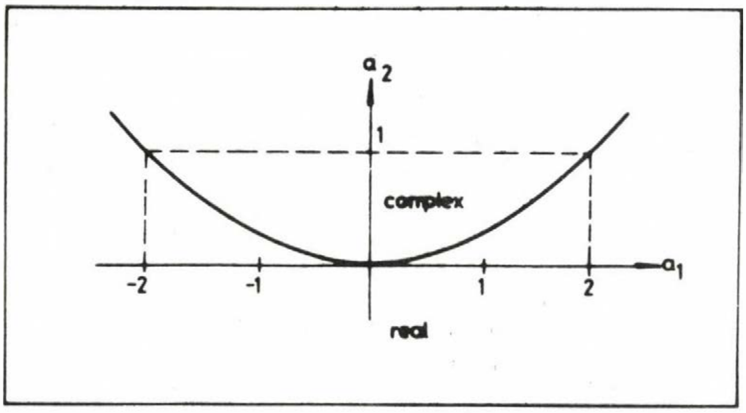

Fig. 3.1-6

For complex poles $r^2 = a_2$, thus the stability is expressed by the condition

$$r = \sqrt{a_2} < 1 \tag{3.1-45}$$

i.e.

$$a_2 < 1 \tag{3.1-46}$$

means the domain of complex stable roots shown in Fig. 3.1-7.

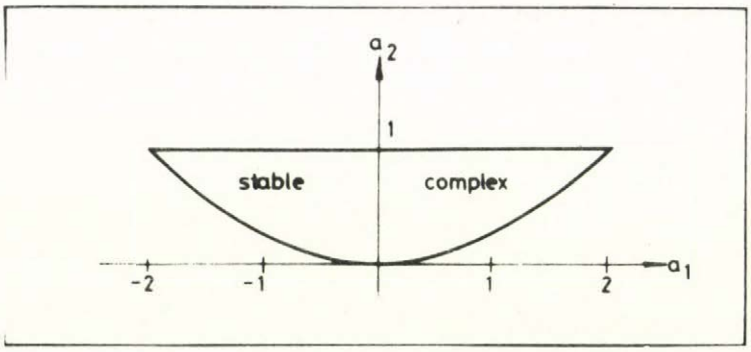

Fig. 3.1-7

As mentioned above, only the right side with a positive real part of the unit circle is considered as a permitted domain by practical considerations. This means that the SHANNON principle will be modified as to claim four samples by periods according to the four parameters of the damped sinus signal. We shall revert to this problem.

According to the above, the condition

$$\mathrm{Re} > 0 \tag{3.1-47}$$

i.e. on the basis of (3.1-42) the condition

$$a_1 < 0 \tag{3.1-48}$$

is obtained for the permitted domain. Fig. 3.1-8 shows the parameter domain permitted.

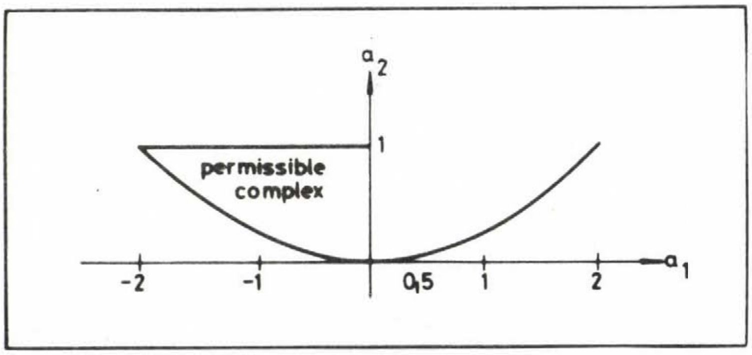

Fig. 3.1-8

Consider now how the parameters of the denominator of the discrete system depend on those of the continuous system and the sampling period.

On the basis of Eq. (3.1-31)

$$a_2 = e^{-\alpha_1 h} = r^2 = e^{-2\xi x} \qquad (3.1\text{-}49)$$

where (3.1-29), is used, as well as the relations

$$\frac{\alpha_1}{\alpha_2} = 2\xi T \quad \text{and} \quad \frac{1}{\alpha_2} = T^2 \qquad (3.1\text{-}50)$$

obtainable from the comparison of the denominators of (3.1-30), the resulting equation is

$$\alpha_1 = 2\xi T\alpha_2 = \frac{2\xi}{T} \quad . \qquad (3.1\text{-}51)$$

a_2 according to (3.1-49) yields therefore the square of the distance of the roots from the origin on the z plane. So that with the step response equivalent transformation

$$r = \sqrt{e^{-2\xi x}} = e^{-\xi x} \quad . \qquad (3.1\text{-}52)$$

Consider now how a_1 changes as a function of x, ξ.

From Eq. (3.1-34)

$$a_1 = -2\sqrt{a_2}\cos h\gamma_i = -2\sqrt{e^{-\alpha_1 h}}\cos h\sqrt{\alpha_2 - \gamma_r^2} =$$

$$= -2\,e^{-\alpha_1 h/2}\cos h\sqrt{\alpha_2 - (\frac{\alpha_1}{2})^2}. \qquad (3.1\text{-}53)$$

Considering now the relations (3.1-50) and (3.1-51)

$$a_1 = -2\,e^{-\frac{2\xi}{T}\frac{h}{2}}\cos h\sqrt{\frac{1}{T^2} - (\frac{\xi}{T})^2} =$$

$$-2\,e^{-\xi\frac{h}{T}}\cos\frac{h}{T}\sqrt{1-\xi^2} = -2\,e^{-\xi x}\cos x\sqrt{1-\xi^2}. \qquad (3.1\text{-}54)$$

Define now the function $a_1(a_2)$ parametizing in ξ.
Expressing x from (3.1-49)

$$x = -\frac{1}{2\xi}\ln a_2 \qquad (3.1\text{-}55)$$

and substituting in (3.1-54)

$$a_1 = -2\,e^{-\xi\left(-\frac{1}{2\xi}\ln a_2\right)}\cos\left[\left(-\frac{1}{2\xi}\ln a_2\right)\sqrt{1-\xi^2}\right] =$$

$$= -2\sqrt{a_2}\cos\left(\frac{\ln a_2}{2}\sqrt{\frac{1-\xi^2}{\xi^2}}\right). \qquad (3.1\text{-}56)$$

This latter can be also written otherwise

$$a_1 = -2\ r \cos\left(\frac{\ln r^2}{2}\sqrt{\frac{1-\xi^2}{\xi^2}}\right) = -2\ r \cos\left(\ln r\sqrt{\frac{1-\xi^2}{\xi^2}}\right). \tag{3.1-57}$$

Let us illustrate the results obtained so far in the various figures enabling graphic construction. Fig. 3.1-9 shows the

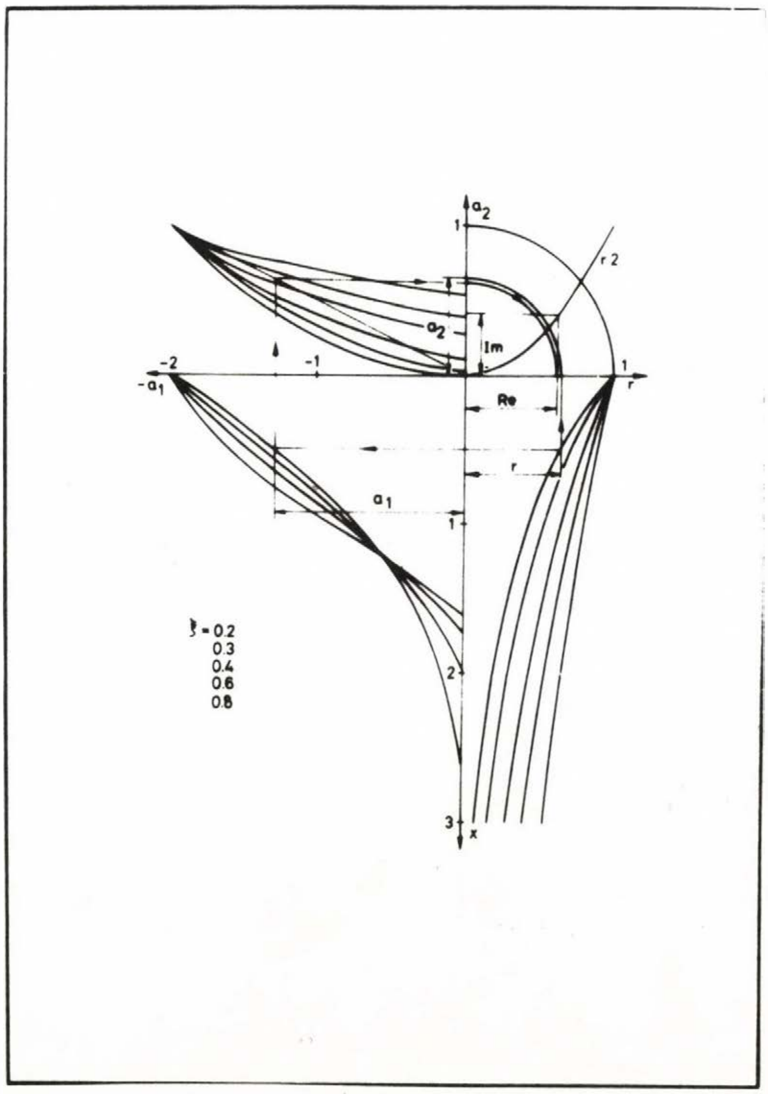

Fig. 3.1-9

location of roots, as well as the change of a_1 and a_2. Obviously both $a_1(x)$ and $r(x)$ curves are parametrized in ξ.

Fig. 3.1-10 shows the character of the function r(x).

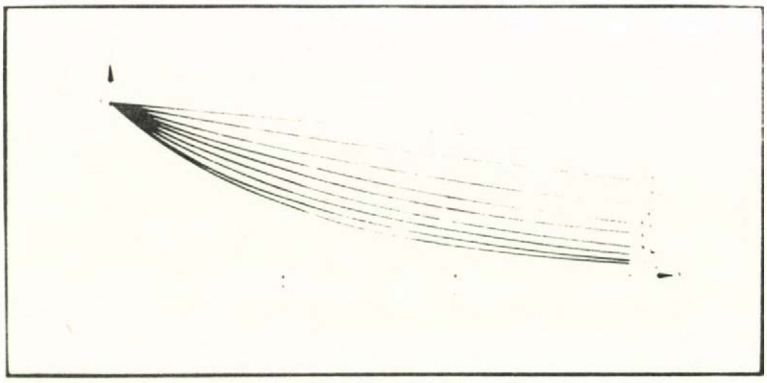

Fig. 3.1-10.

Fig. 3.1-9 produces the related values a_1 and a_2. While we presented in this figure the construction of the pole, in Fig. 3.1-11 we present the change of the coefficients of the denominator polynomials.

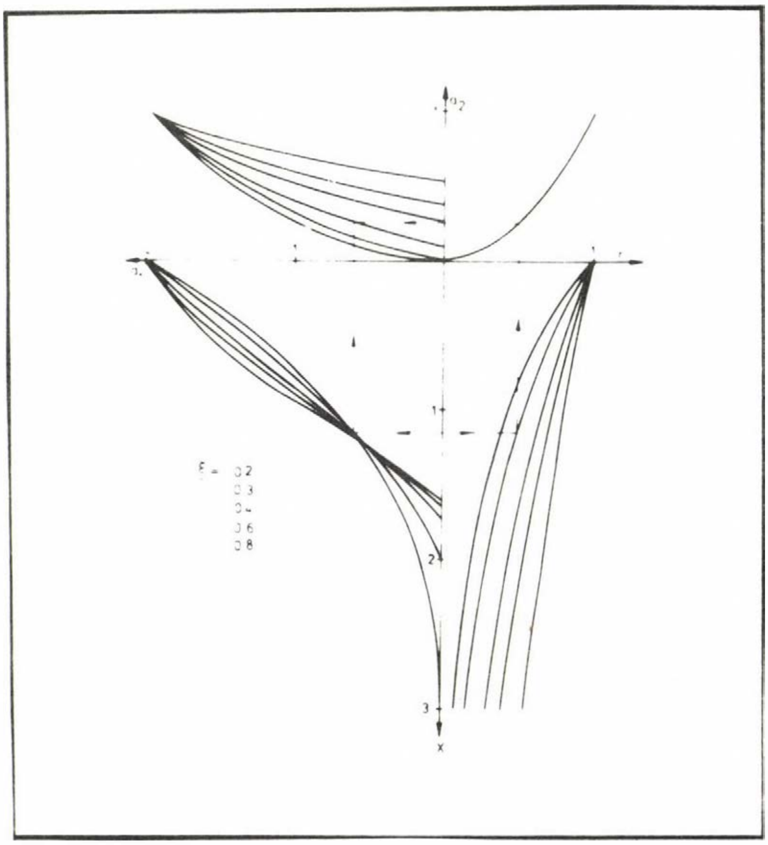

Fig. 3.1-11.

On the basis of the formulae (3.1-49) and (3.1-54), the curves $a_2(x)$-ξ and $a_1(x)$-ξ can also be drawn, cf. Figs. 3.1-12 and 3.1-13.

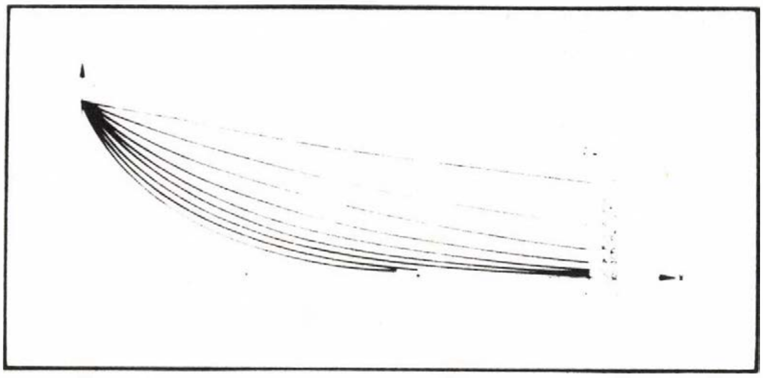

Fig. 3.1-12

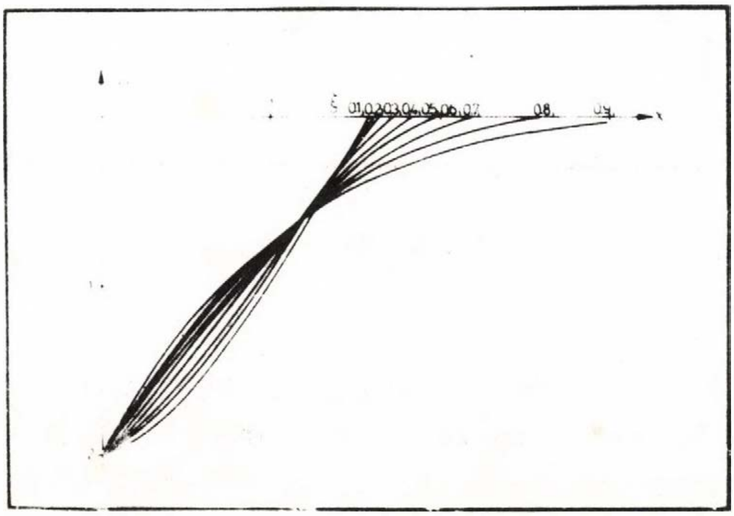

Fig. 3.1-13

Likewise the function $a_1(a_2)$-ξ can also be represented parametrized in ξ (Fig. 3.1-14).

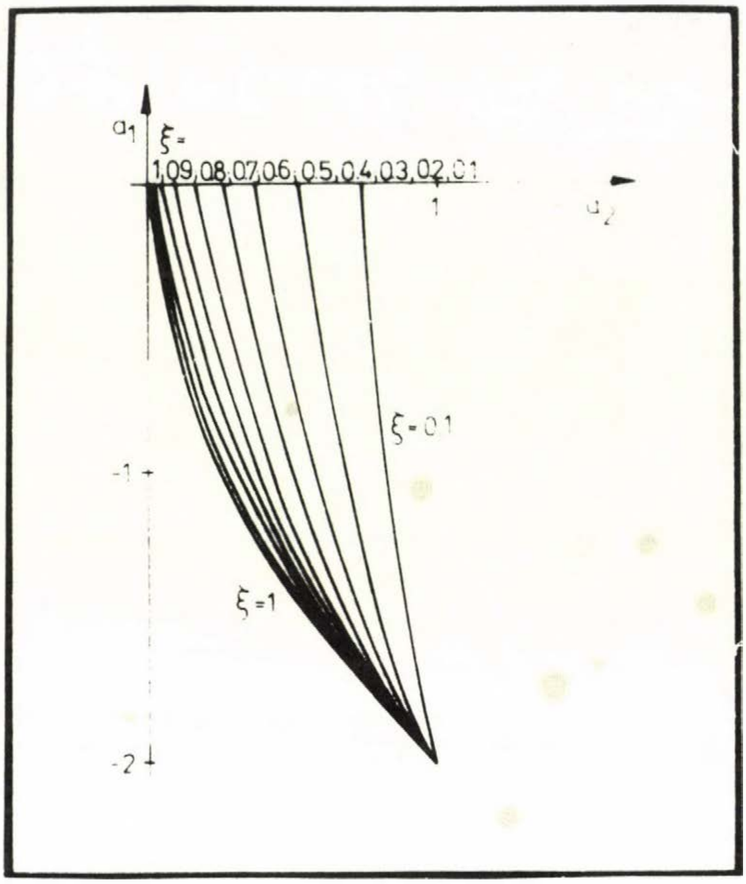

Fig. 3.1-14

So far x = h/T was used as relative sampling time, i.e. it was related to the time constant corresponding to the cutting frequency. Also the time period of the damped sine of an oscillating term can be used as a reference base. Compute on the basis of (3.1-30) the poles of the continuous system

$$s_{1,2} = \frac{-2\xi T \pm \sqrt{4\xi^2 T^2 - 4T^2}}{2 T^2} = -\frac{\xi}{T} \pm j \frac{1}{T} \sqrt{1-\xi^2} .$$

Fig. 3.1-15 shows the location of the roots. On the basis of

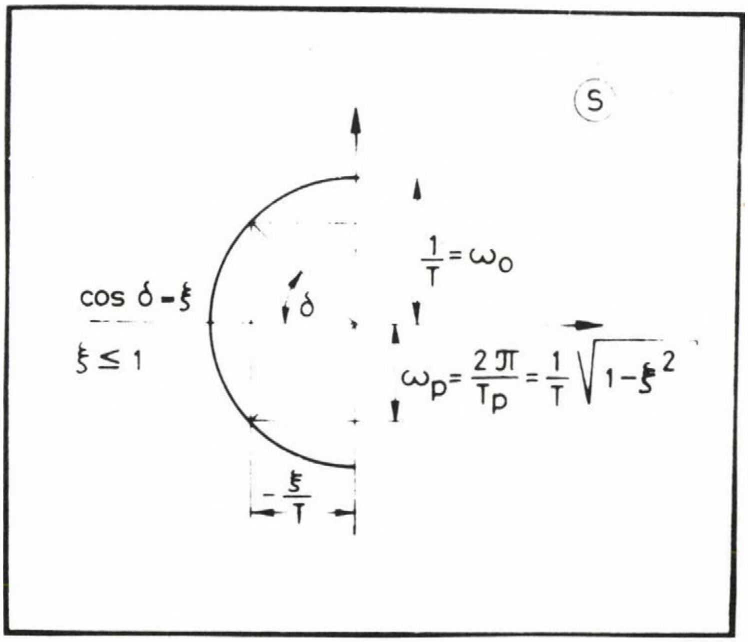

Fig. 3.1-15.

the relations of the figure, the frequency of the oscillation

$$\omega_p = \text{Im} = \frac{1}{T} \sqrt{1 - \xi^2} = \omega_o \sqrt{1 - \xi^2} = \frac{2\pi}{T_p} \qquad (3.1\text{-}58)$$

and its time period

$$T_p = \frac{2\pi}{\omega_p} = \frac{2\pi}{\sqrt{1 - \xi^2}} T. \qquad (3.1\text{-}59)$$

By introducing this way the relative sampling rate $y = h/T_p$

$$y = \frac{h}{T_p} = h \frac{\sqrt{1 - \xi^2}}{2\pi} \cdot \frac{1}{T} = \frac{h}{T} \frac{\sqrt{1 - \xi^2}}{2\pi} = \frac{\sqrt{1 - \xi^2}}{2\pi} x. \qquad (3.1\text{-}60)$$

Express now the previously obtained important relations by using the substitution

$$x = \frac{2\pi}{\sqrt{1-\xi^2}}\, y \,. \tag{3.1-61}$$

Thus

$$a_2 = r^2 = e^{-2\xi x} = e^{-\frac{4\pi\xi}{\sqrt{1-\xi^2}}\,y} \tag{3.1-62}$$

i.e.

$$r = e^{\frac{-2\pi\xi}{\sqrt{1-\xi^2}}\,y} \,. \tag{3.1-63}$$

Further

$$a_1 = -2e^{-\xi x}\cos x\sqrt{1-\xi^2} = -2\,e^{-\frac{2\pi\xi}{\sqrt{1-\xi^2}}\,y}\cos 2\pi y \,. \tag{3.1-64}$$

As the sampling is carried out according to Fig. 3.1-16, it

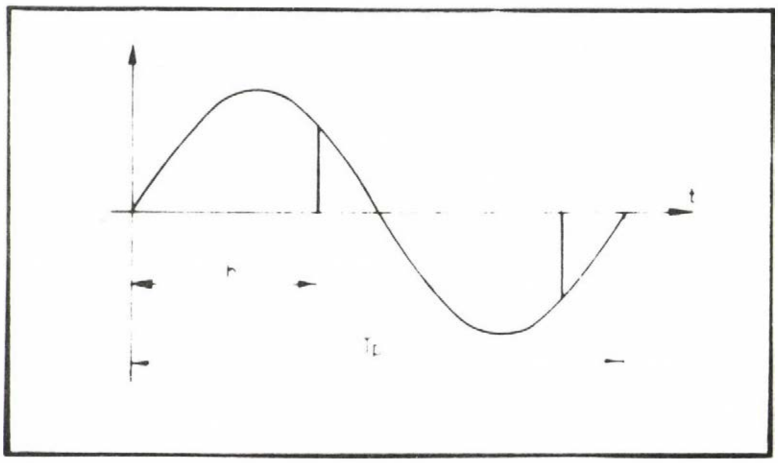

Fig. 3.1-16

can be seen that y gets an upper bound from the sampling principle of SHANNON (also from h for x, of course). Accordingly, the condition

$$h \leqq \frac{T_p}{2} \,, \tag{3.1-65}$$

or the corresponding

$$x = \frac{h}{T} \leqq \frac{T_p}{2} = \frac{\pi}{\sqrt{1-\xi^2}} \tag{3.1-66}$$

or the inequality

$$y = \frac{h}{T_p} \leqq 0.5 \tag{3.1-67}$$

has to be fulfilled, i.e. at least one sample should be taken for every half-period.

Also another constrain is taken into consideration for the sampling time. As discussed above, a_1 cannot be positive for some reasons, i.e. the roots must fall into the right side half of the unit circle. According to (3.1-64) it is required that $\cos 2\pi y \geqq 0$. The domain permitted by (3.1-67) is limited by this latter condition to the domain

$$y \leqq 0.25 \quad . \tag{3.1-68}$$

This corresponds to the requirement that we should take at least four samples from one period of the damping sine signal in order to determine the complex conjugate root pair (4 parameters). The bound (3.1-66) relating to x accordingly changes to the condition

$$x \leqq \frac{\pi}{2\sqrt{1-\xi^2}} = x_{max}(\xi) , \qquad (3.1\text{-}69)$$

where the upper bound of the sampling rate is the function of ξ. Fig. 3.1-17 shows this bound in the function of ξ.

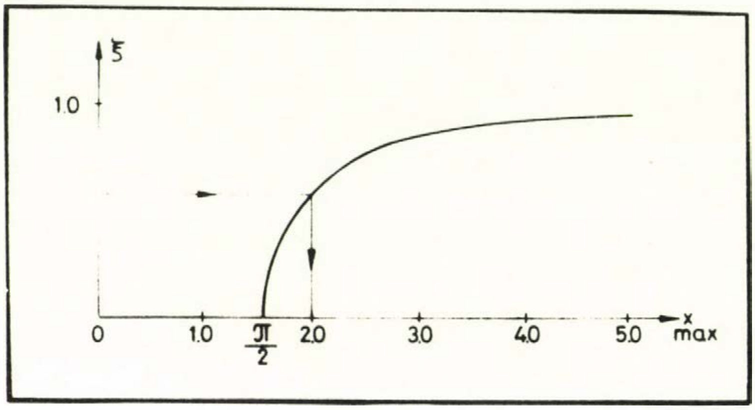

Fig. 3.1-17.

Fig. 3.1-18 shows the relation $y = y(x)$ parametrizing in ξ.

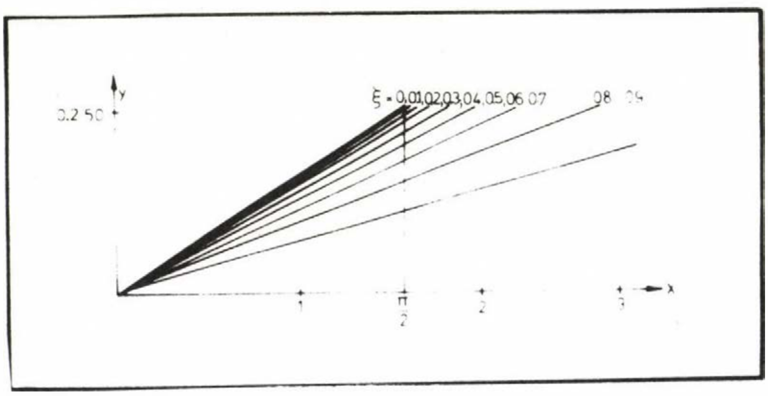

Fig. 3.1-18

We can see the dependence of a_2, r and a_1 on y in Figs. 3.1-19, 3.1-20 and 3.1-21, respectively.

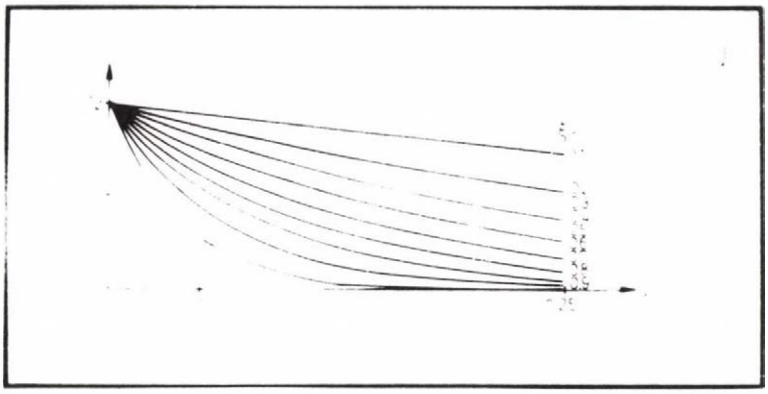

Fig. 3.1-19

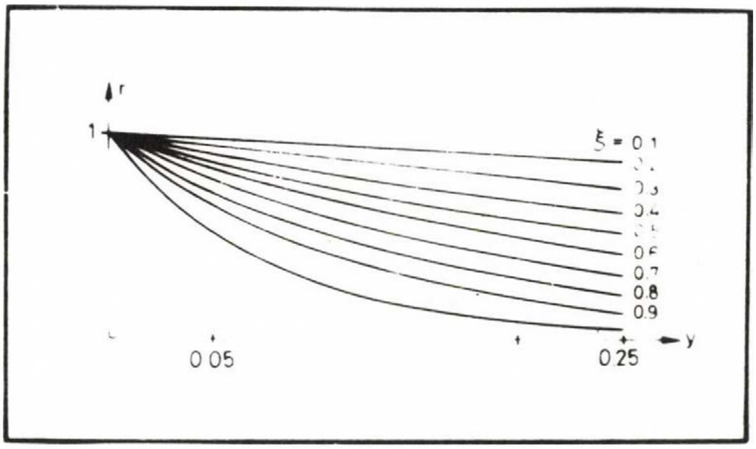

Fig. 3.1-20

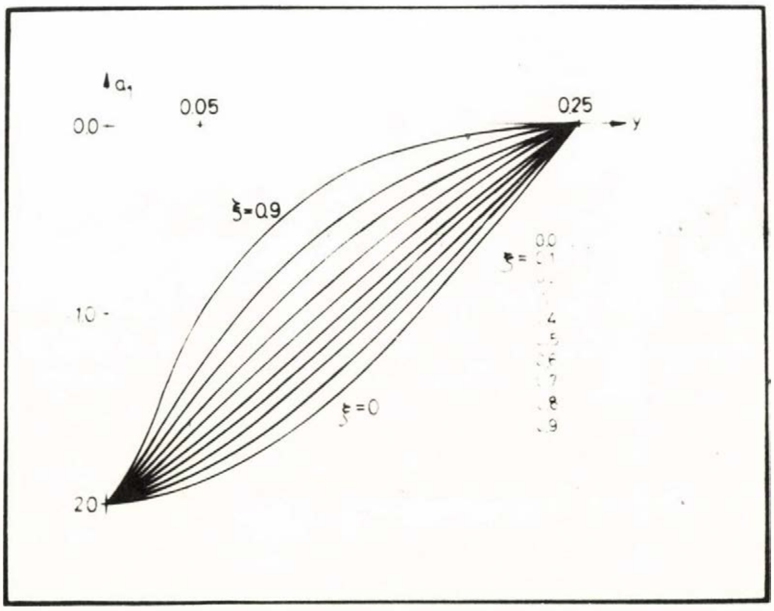

Fig. 3.1-21

We can, of course, plot even now the complex figure corresponding to Fig. 3.1-11, but now for y (cf. Fig. 3.1-22).

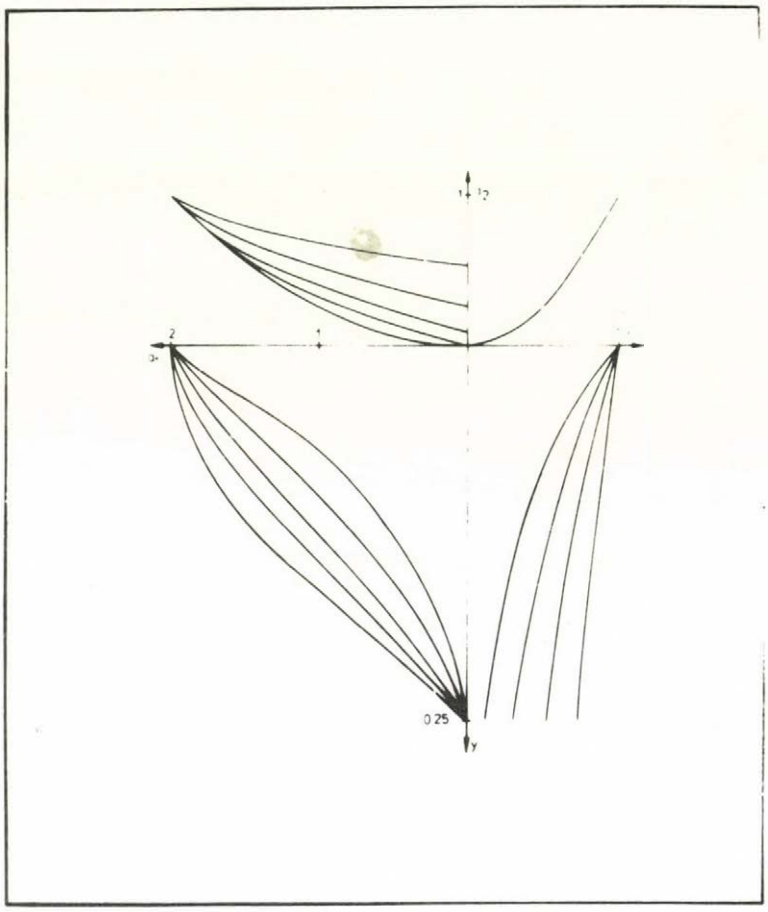

Fig. 3.1-22

Similarly to the first-order system investigate now the possibility of the optimization of the sampling time. A complex conjugate pole pair is investigated in the plane (s) and (z) (Cf. Fig. 3.1-23).

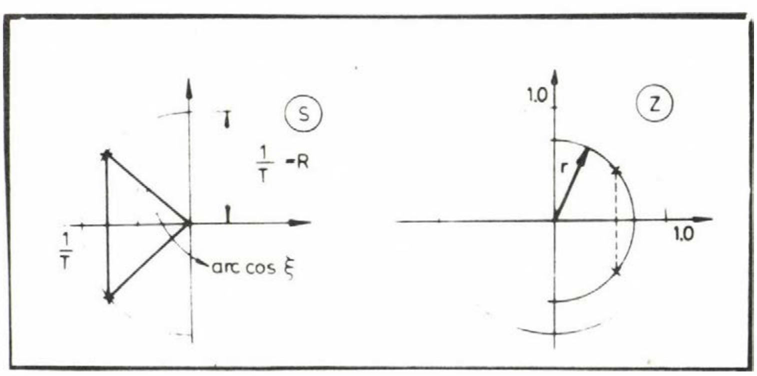

Fig. 3.1-23

The pole pair located in the circle with radius R in the plane (s) should be located in the circle with radius r in the plane (z) . (The circles denote here only geometrical sizes without referring to the geometrical places of the transformation). Moreover

$$R = \frac{1}{T} \qquad \text{and} \qquad r = \sqrt{a_2} \tag{3.1-70}$$

and between these two quantities on the basis of (3.1-52), the relation is

$$R = -\frac{1}{\xi h} \ln r \quad . \tag{3.1-71}$$

As against the first-order case instead of the roots, we consider now the distance measured from the origin as the variable of the sensitivity problem. For this task therefore the sensitivity function

$$E = \frac{\left| \frac{\partial R}{\partial r} \right|}{R} \tag{3.1-72}$$

can be defined, or the appropriate insensitivity function is

$$Q = \frac{1}{E} = \frac{R}{\left| \frac{\partial R}{\partial r} \right|} \quad . \tag{3.1-73}$$

Let us determine the partial derivative in the relation

$$\frac{\partial R}{\partial r} = \frac{\partial}{\partial r} \left(- \frac{1}{\xi h} \ln r \right) = - \frac{1}{\xi h} \, \frac{1}{r} \; . \tag{3.1-74}$$

Accordingly

$$Q = Q(x) = \frac{R}{\frac{1}{\xi h} \ \frac{1}{r}} = \xi\, h\, r\, R = \xi\, h\, R\, e^{-\xi x} =$$

$$= \xi\, h \frac{1}{T} e^{-\xi x} = \xi\, x\, e^{-\xi x} \, . \tag{3.1-75}$$

Determine the maximum of the insensitivity function by **x**. The condition of the extremum value for this is

$$\frac{\partial Q(x)}{\partial x} = \xi\, e^{-\xi x} + \xi\, x \left(- \xi\, e^{-\xi x} \right) = 0 \tag{3.1-76}$$

i.e.

$$1 - \xi\, x = 0 \, . \tag{3.1-77}$$

Hence the optimal x

$$x_{opt} = \frac{1}{\xi} \tag{3.1-78}$$

i.e.

$$h_{opt} = \frac{T}{\xi} \quad . \tag{3.1-79}$$

(It is easy to understand from the second derivative that the extremum value is a maximum). On the basis of (3.1-60) the optimal value of y

$$y_{opt} = \frac{\sqrt{1-\xi^2}}{2\pi} \quad x_{opt} = \frac{1}{2\pi\xi}\sqrt{1-\xi^2} = \frac{1}{2\pi}\sqrt{\frac{1}{\xi^2}-1} \tag{3.1-80}$$

As the condition $y \leqq 0.25$ must hold, only that y_{opt} value can be accepted for which the condition

$$y_{opt} = \frac{1}{2\pi}\sqrt{\frac{1}{\xi^2}-1} \leqq 0.25 \tag{3.1-81}$$

is fulfilled, whence we obtain that

$$\xi \geqq \frac{1}{\sqrt{1+\frac{\pi^2}{4}}} = \xi_{min} = 0.53703 \; . \tag{3.1-82}$$

Optimal sampling can be carried out only for these ξ values. In the cases $\xi<\xi_{min}$ we have to be satisfied with $y^{*}_{opt} = 0.25$ and the corresponding sampling rate

$x^{*}_{opt} = \frac{\pi}{2\sqrt{1-\xi^2}}$. This is presented on Fig. 3.1-24. Fig. 3.1-25

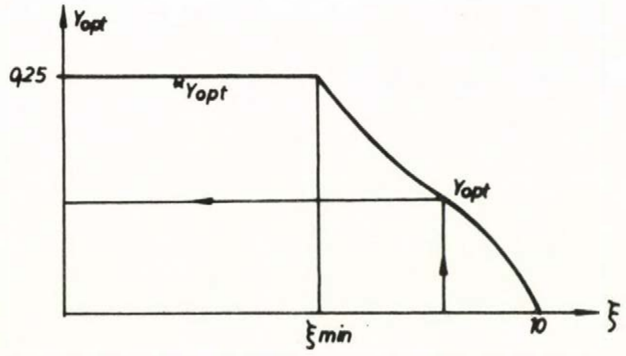

Fig. 3.1-24

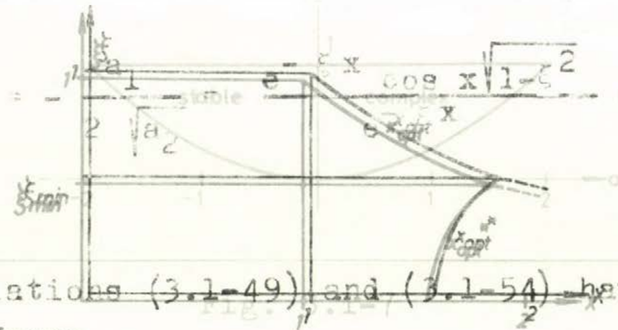

Fig. 3.1-25

shows the value of x_{opt} as a function of ξ. Here the following relations are used:

$$x^{*\xi}_{opt} = \frac{\pi}{2\sqrt{1-\xi^2}} ; \qquad \xi < \xi_{min} \tag{3.1-83}$$

and

$$x_{opt} = \frac{1}{\xi} , \tag{3.1-84}$$

further

$$x^{*}_{opt}(\xi = 0) = \frac{\pi}{2} = 1.5708 \tag{3.1-85}$$

and

$$x^{*}_{opt}(\xi = \xi_{min}) = 1.8621 . \tag{3.1-86}$$

Similarly to the first-order system, we have presented in Fig. 3.1-26 the insensitivity function Q(x). Thus even now

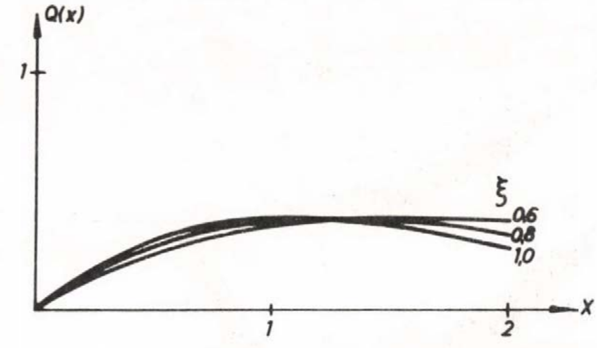

Fig. 3.1-26

we can represent alike Fig. 3.1-5 the determination of the poles with optimal location, cf. Fig. 3.1-27. It is more

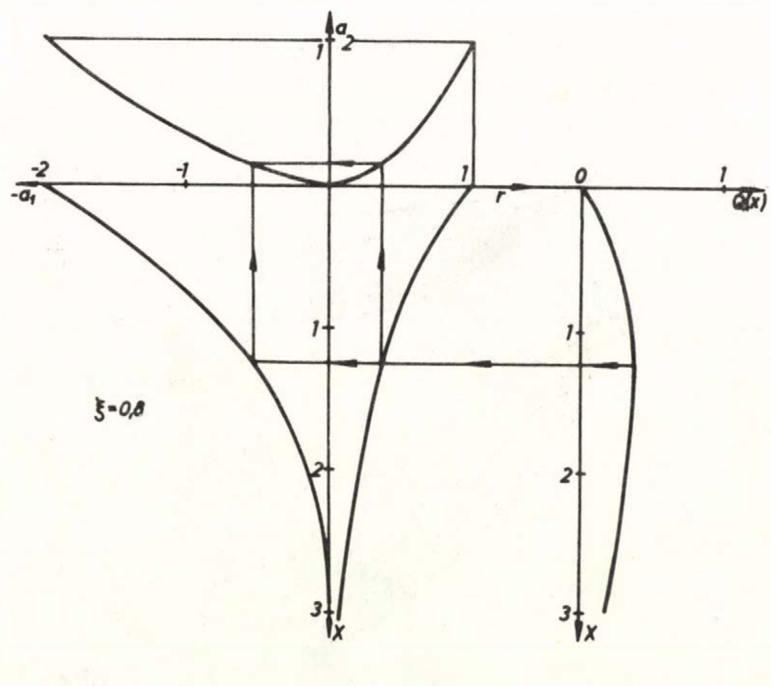

Fig. 3.1-27

practical to build together the figure instead of the function $Q(x)$ with the figure (3.1-25). This combined set of curves can be seen in Fig. 3.1-28. Actually Fig. 3.1-29, too, is of similar construction, but for the relative sampling rate y.

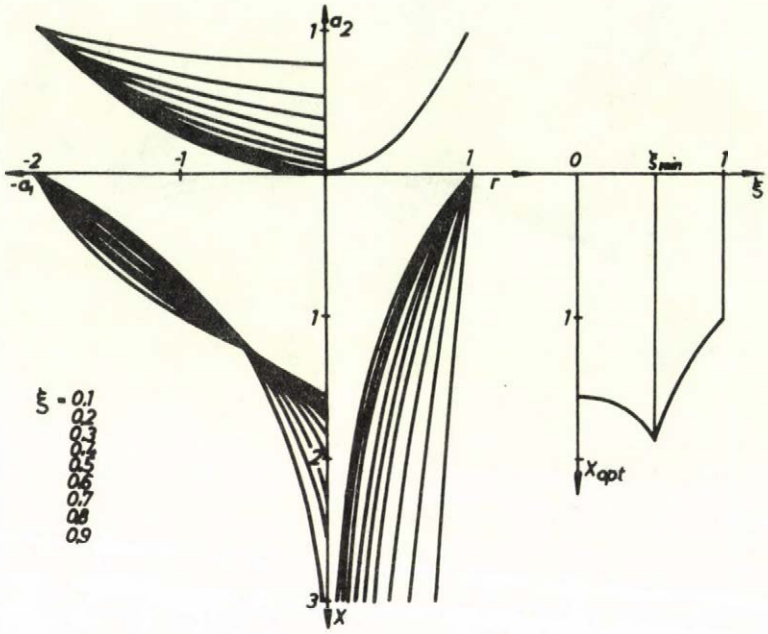

Fig. 3.1-28

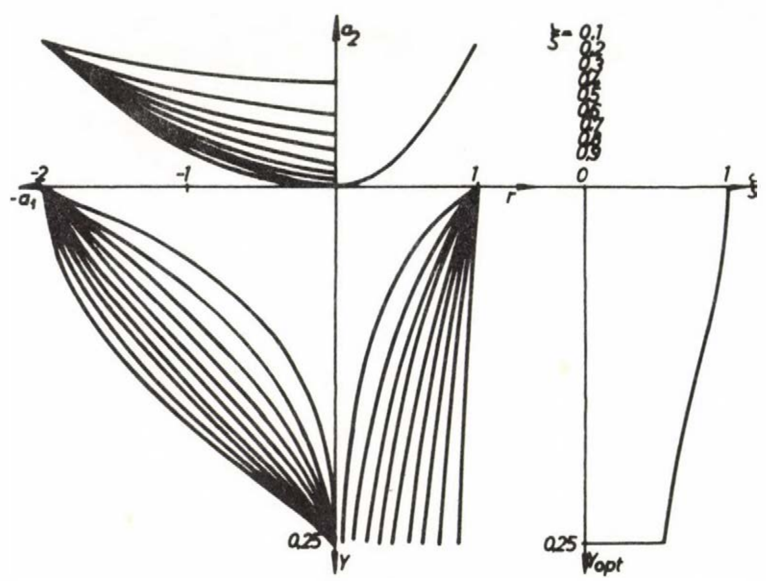

Fig. 3.1-29

(Note that if according to the SHANNON principle, we would allow for y the inside of the left half-circle, too: $y \leq 0.5$, then $\xi_{min} = \frac{1}{\sqrt{1+\pi^2}} = 0.3033$ would result, which is something less, but neither in this case could the optimal sampling comprehend the whole ξ domain.)

Consider now the optimization of the angles giving the location of the poles by the sampling period. For this purpose consider Fig. 3.1-30.

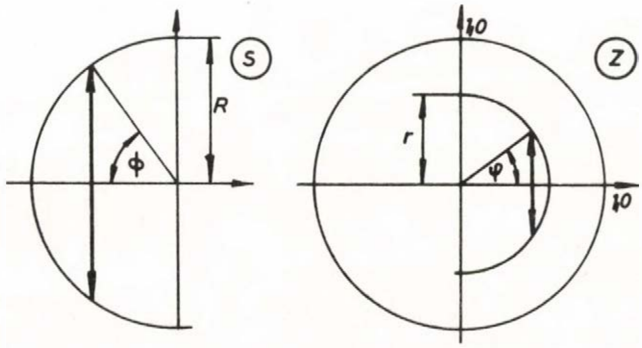

Fig. 3.1-30

The insensitivity function for this case is

$$Q \triangleq \frac{|\Phi|}{\left|\frac{\partial \Phi}{\partial \varphi}\right|} , \tag{3.1-87}$$

and the sensitivity function

$$E \triangleq \frac{1}{Q} = \frac{\left|\frac{\partial \Phi}{\partial \varphi}\right|}{|\Phi|} . \tag{3.1-88}$$

From the comparison of Figs. 3.1-30 and 3.1-23 we obtain

$$\Phi = \text{arc cos } \xi . \tag{3.1-89}$$

Further on the basis of (3.1-39) and (3.1-42)

$$\cos\varphi = \frac{-\frac{a_1}{2}}{\sqrt{a_2}} = -\frac{a_1}{2\sqrt{a_2}} = \frac{e^{-\xi x}\cos x\sqrt{1-\xi^2}}{e^{-\xi x}} = \cos x\sqrt{1-\xi^2}, \tag{3.1-90}$$

where the relations (3.1-49) and (3.1-54) have already been considered. Hence

$$\varphi = x\sqrt{1-\xi^2} \tag{3.1-91}$$

i.e.

$$\xi = \frac{1}{x}\sqrt{x^2-\varphi^2} = \sqrt{1-\left(\frac{\varphi}{x}\right)^2} \tag{3.1-92}$$

Considering this in (3.1-89), we obtain

$$\Phi = \arccos\sqrt{1-\left(\frac{\varphi}{x}\right)^2} . \tag{3.1-93}$$

Calculate now the partial derivative in (3.1-87):

$$\frac{\partial\Phi}{\partial\varphi} = -\frac{1}{\sqrt{1-1-\left(\frac{\varphi}{x}\right)^2}}\,\frac{1}{2}\,\frac{1}{\sqrt{1-\left(\frac{\varphi}{x}\right)^2}}\left(\frac{-2}{x^2}\right)\varphi =$$

$$= \frac{x}{\varphi}\,\frac{1}{\sqrt{1-\left(\frac{\varphi}{x}\right)^2}}\,\frac{\varphi}{x^2} = \left.\frac{1}{x\sqrt{1-\left(\frac{\varphi}{x}\right)^2}}\right|_{\varphi=x\sqrt{1-\xi^2}} =$$

$$= \frac{1}{x\sqrt{1-\left(\frac{x\sqrt{1-\xi^2}}{x}\right)^2}} = \frac{1}{x\sqrt{1-\left(1-\xi^2\right)}} = \frac{1}{x\xi} \quad . \qquad (3.1\text{-}94)$$

Thus

$$\frac{|\Phi|}{\left|\frac{\partial \Phi}{\partial \varphi}\right|} = \frac{\text{arc cos } \xi}{\frac{1}{x\xi}} = x\,\xi \text{ arc cos } \xi \; . \qquad (3.1\text{-}95)$$

The insensitivity function obviously has its maximum in $x = \infty$ (cf. Fig. 3.1-31). Accordingly the greater x, the better, to which a large φ corresponds.

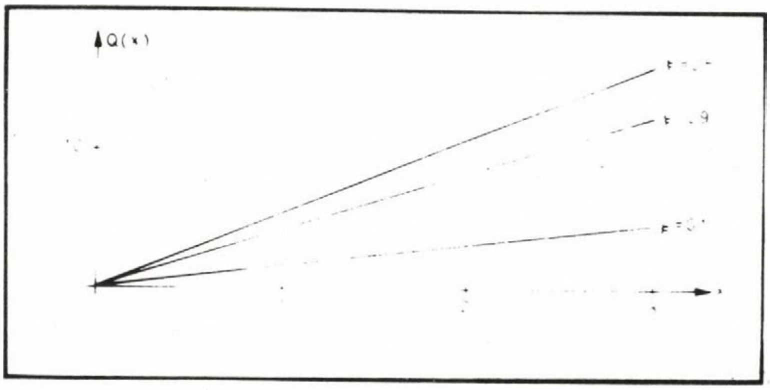

Fig. 3.1-31

This means that from the two cases the better is where the poles are farer from the real axis.

Construct now an insensitivity function taking into account both radial and tangential insensitivity. From (3.1-75)

$$Q_R(x) = \xi x \, e^{-\xi x} \qquad (3.1\text{-}96)$$

and from (3.1-95)

$$Q_{\Phi}(x) = x\xi \arccos\xi . \qquad (3.1\text{-}97)$$

Constructing the joint sensitivity function

$$Q_{R\Phi}(x) = Q_R(x)\, Q_{\Phi}(x) = \xi\, x\, e^{-\xi x}\, x\xi \arccos\xi = x^2 \xi^2\, e^{-\xi x} \arccos\xi . \qquad (3.1\text{-}98)$$

its maximum (see Fig. 3.1-32) is ensured by the value $x_{opt} = 2/\xi$.

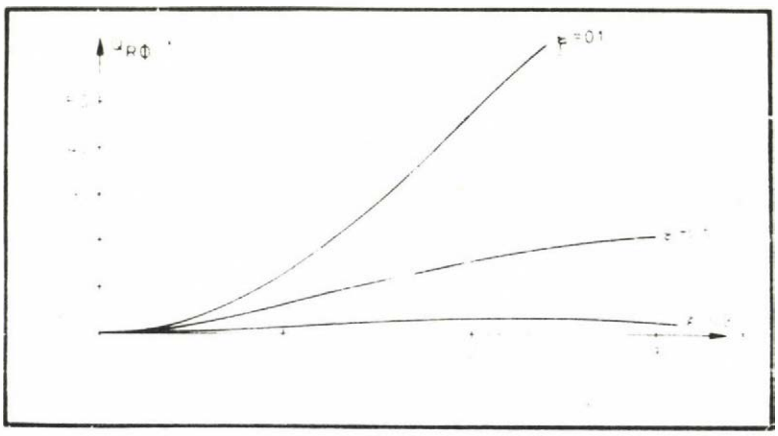

Fig. 3.1-32

Also this investigation refers to the problem which will thereinafter return several times, viz, the strong dependence of the sampling time on the criterion chosen. (Besides neither the behaviour of the denominator characterizes unanimously the optimization of the whole system, thus it will at any rate be necessary to study also the insensitivity of the numerator, consequently of the zeros.)

On basis of the results obtained for the optimal sampling time the optimal value of coefficients a_1 and a_2 can also be determined.

The optimal value of x

$$x_{opt} = \begin{cases} \frac{1}{\xi}, \text{ if } \xi > \xi_{min} = \frac{2}{\sqrt{4+\pi^2}} ; \quad \xi < 1 \\ \\ \frac{\pi}{2\sqrt{1-\xi^2}}, \text{ if } \xi < \xi_{min} ; \quad \xi > 0 \end{cases} \tag{3.1-99}$$

and of y

$$y_{opt} = \begin{cases} \frac{\sqrt{1-\xi^2}}{2\pi\xi}, \text{ if } \xi > \xi_{min} \\ \\ 0.25 \quad , \text{ if } \xi < \xi_{min} \end{cases} \tag{3.1-100}$$

Accordingly

$$a_{2\ opt} = \begin{cases} e^{-2} = 0.1353, \text{ if } \xi > \xi_{min} \\ \\ e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} \quad \text{ if } \xi < \xi_{min} \end{cases} \tag{3.1-101}$$

further by expressing also the value of a_1 by the optimal x:

$$a_{1\,opt} = \begin{cases} -2e^{-1} \cos \sqrt{\dfrac{1-\xi^2}{\xi^2}} = -\,0.7358 \cos \sqrt{\dfrac{1-\xi^2}{\xi^2}} \\ \\ 0 \end{cases} \tag{3.1-102}$$

Fig. 3.1-33 shows the relations (3.1-101) and (3.1-102). By

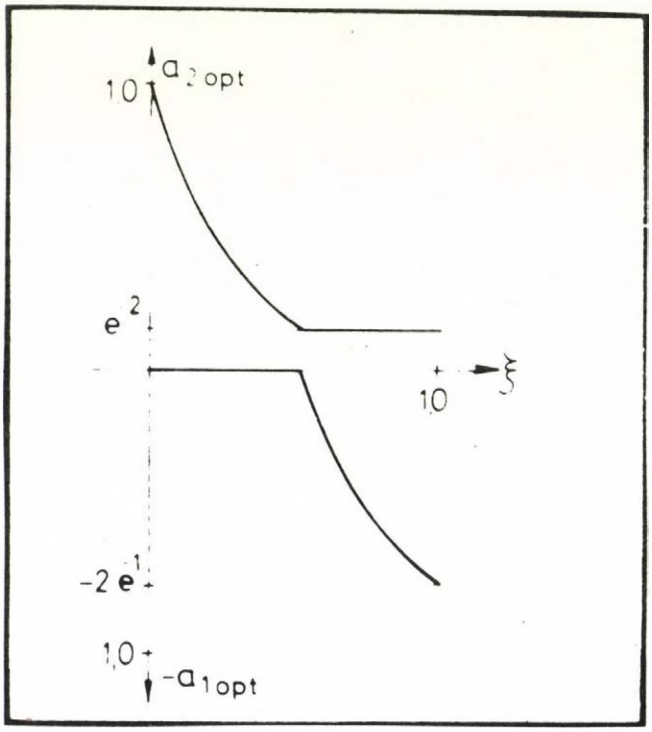

Fig. 3.1-33

plotting the above relations on the plane a_1, a_2 we can denote the location of these coefficients at the time of optimal sampling, cf. Fig. 3.1-34.

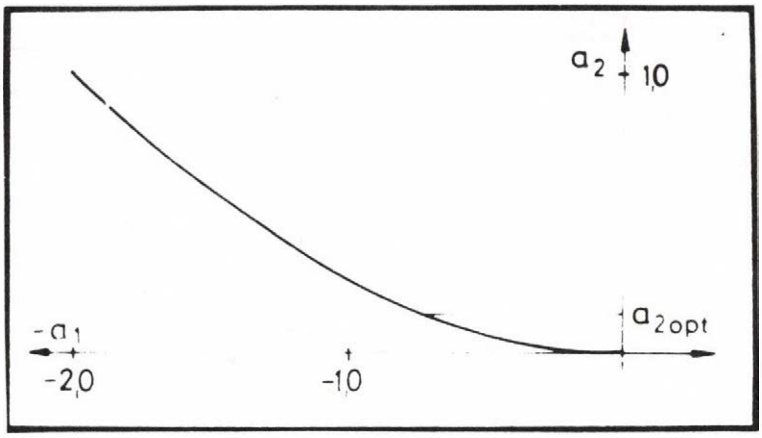

Fig. 3.1-34

Consider now a general case when there are several disjunct real poles and complex pole pairs. Let in this case the sensitivity function be

$$E \triangleq \frac{\prod_{i=1}^{n} \left|\frac{\partial s_i}{\partial z_i}\right| \prod_{j=1}^{2m} \left|\frac{\partial R_j}{\partial r_j}\right|}{\frac{1}{T_o}} , \qquad (3.1\text{-}103)$$

where $\underline{n}$ is the number of the real poles, m the number of the complex pole pairs. T_o is the appropriately chosen reference value. (see later). In the numerator there are the products of the sensitivity derivatives used for first- and second-order systems. The insensitivity function is

$$Q = \frac{1}{E} = \frac{\frac{1}{T_o}}{\prod_{i=1}^{n} \left|\frac{\partial s_i}{\partial z_i}\right| \prod_{j=1}^{2m} \left|\frac{\partial R_j}{\partial r_j}\right|} . \qquad (3.1\text{-}104)$$

Determine the partial derivatives:

$$\left|\frac{\partial s_i}{\partial z_i}\right| = \frac{e^{h\alpha_i}}{h} \qquad (3.1\text{-}105)$$

and

$$\left| \frac{\partial R_j}{\partial r_j} \right| = \frac{e^{h\alpha_j \xi_j}}{h \xi_j} \; . \tag{3.1-106}$$

This way the sensitivity function is

$$E(h) = \frac{\prod_{i=1}^{n} \frac{e^{h\alpha_i}}{h} \prod_{j=1}^{2m} \frac{e^{h\alpha_j \xi_j}}{h \xi_j}}{\frac{1}{T_o}} =$$

$$= h^{-n} h^{-2m} T_o \, \xi_j \, e^{h \left(\sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{2m} \alpha_j \xi_j \right)} =$$

$$= T_o \, h^{-(n+2m)} \, e^{h\gamma} \, , \tag{3.1-107}$$

where the quantity

$$\gamma = \sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{2m} \alpha_j \xi_j \tag{3.1-108}$$

have been introduced.

For the minimization of E

$$\frac{\partial E}{\partial h} = T_o \left[\frac{-(n+2m)}{h^{(n+2m+1)}} \; e^{h\gamma} + \frac{1}{h^{(n+2m)}} \; \gamma e^{h\gamma} \right] = 0$$

whence

$$-(n+2m) + \gamma h = 0$$

and

$$h_{opt} = \frac{n+2m}{\gamma} = \frac{1}{\frac{\gamma}{n+2m}} = \frac{1}{\frac{1}{T_o}} = T_o \; . \tag{3.1-109}$$

In an analogous way to the above, here, for practical reasons, the average, i.e. the medium-frequency time constant

$$T_o = \left(\frac{\gamma}{n+2m}\right)^{-1} = \frac{n+2m}{\gamma} = \frac{n+2m}{\sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{2m} \alpha_j \xi_j} \tag{3.1-110}$$

have been introduced.

By this choice the optimal relative sampling time became formally again the value

$$x_{opt} = \frac{h_{opt}}{T_o} = 1 \tag{3.1-111}$$

As

$$\alpha_j \xi_j = \frac{1}{T_j} \xi_j = \mathrm{Re}\, s_j \tag{3.1-112}$$

thus

$$\frac{1}{T_o} = \frac{\sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{2m} \alpha_j \xi_j}{n + 2m} = \frac{\sum_{i=1}^{n+2m} \mathrm{Re}\ s_i}{n + 2m} = \frac{\sum_{i=1}^{n+2m} s_i}{n + 2m}, \tag{3.1-113}$$

i.e. the arithmetical mean of (the real part of) the poles.

We can, of course, apply also now various weightings, i.e. calculate the "medium frequency time constant" in the form

$$\frac{1}{T_o} = \frac{\sum_{i=1}^{n+2m} w_i \ \mathrm{Re}\ s_i}{n + 2m} \tag{3.1-114}$$

where w_i-s are the various weights. For practical reasons w_i-s are proportional with the residues and

$$\sum_{i=1}^{n+2m} w_i = 1.$$

The result obtained for the first order system is trivially in accordance with the general one according to (3.1-109), but this similarly holds for the second order system, too. It follows, namely, from (3.1-79)

$$h_{opt} = \frac{T}{\xi} = \frac{1}{\xi \frac{1}{T}} = \frac{1}{\mathrm{Re}\ s} .$$

Investigate now the sensitivity of the poles of the continuous second order system with respect to the uncertainty of the parameters in the denominator of the identified discrete transfer function. (Now only deterministic relations are discussed.) Let the transfer function of the discrete system - according to (3.1-29) - be

$$G(z) = \frac{b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{c_1 z^{-1}}{1 - z_1 z^{-1}} + \frac{c_2 z^{-1}}{1 - z_2 z^{-1}} \qquad (3.1\text{-}115)$$

and the step response equivalent continuous system

$$H(s) = \frac{\beta_1 s + \beta_2}{s^2 + \alpha_1 s + \alpha_2} = K \frac{1 + sT_1}{1 + 2\xi T s + T^2 s^2}$$

$$= \frac{c_1}{s - s_1} + \frac{c_2}{s - s_2} \;. \qquad (3.1\text{-}116)$$

The denominator of the discrete system can be expressed with the parameters of the continuous system

$$a_1 = -z_1 - z_2 = -e^{s_1 h} - e^{s_2 h} \;,$$

$$a_2 = e^{(s_1 + s_2) h} \;. \qquad (3.1\text{-}117)$$

Here the relation (3.1-6) and its preliminiaries have been used by considering those statements of Chapter 2 according to which these transformational relations could be applied for complex roots, too. Investigate now the effect of the changes Δa_1, Δa_2 around a given operating point of the parameters a_1, a_2 of the discrete system on the poles of the continuous system. By taking only the first order changes into account

$$\Delta \underline{s} = \begin{bmatrix} \Delta s_1 \\ \\ \Delta s_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial s_1}{\partial a_1} & \frac{\partial s_1}{\partial a_2} \\ \\ \frac{\partial s_2}{\partial a_1} & \frac{\partial s_2}{\partial a_2} \end{bmatrix} \cdot \begin{bmatrix} \Delta a_1 \\ \\ \Delta a_2 \end{bmatrix} = \underline{\underline{J}}(\underline{s}, \underline{a}) \; \Delta \underline{a}. \tag{3.1-118}$$

Here $\underline{\underline{J}}(\underline{s},\underline{a})$ is the corresponding Jacobian matrix for the vectors

$$\underline{s} = \begin{bmatrix} s_1, & s_2 \end{bmatrix}^T$$

and

$$\underline{a} = \begin{bmatrix} a_1, & a_2 \end{bmatrix}^T \quad ,$$

furthermore Δs and Δa denote the changes of the above vectors. The **relation** of the equation (3.1-118) to the sensitivity functions applied heretofore can be presented with the similarly defined sensitivity matrix $\underline{\underline{E}}$:

$$\underline{\underline{E}} = \begin{bmatrix} \frac{1}{|s_1|} \frac{\partial s_1}{\partial a_1} & \frac{1}{|s_1|} \frac{\partial s_1}{\partial a_2} \\ \\ \frac{1}{|s_2|} \frac{\partial s_2}{\partial a_1} & \frac{1}{|s_2|} \frac{\partial s_2}{\partial a_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{|s_1|} & 0 \\ \\ 0 & \frac{1}{|s_2|} \end{bmatrix} \begin{bmatrix} \frac{\partial s_1}{\partial a_1} & \frac{\partial s_1}{\partial a_2} \\ \\ \frac{\partial s_2}{\partial a_1} & \frac{\partial s_2}{\partial a_2} \end{bmatrix} =$$

$$= \underline{\underline{S}}^{-1} \, \underline{\underline{J}} \, (\underline{s},\underline{a}) \; . \qquad (3.1\text{-}119)$$

Here

$$\underline{\underline{S}} = \text{diag} \; \langle |s_1|, \; |s_2| \rangle \; . \qquad (3.1\text{-}120)$$

The insensitivity matrix is accordingly

$$\underline{\underline{Q}} = \underline{\underline{J}} \, (\underline{a},\underline{s}) \; \underline{\underline{S}} \neq \underline{\underline{E}}^{-1} \; . \qquad (3.1\text{-}121)$$

Define now the matrix $\underline{\underline{J}} \, (\underline{a},\underline{s})$:

$$\underline{\underline{J}}(\underline{a},\underline{s}) = \begin{bmatrix} \dfrac{\partial a_1}{\partial s_1} & \dfrac{\partial a_1}{\partial s_2} \\ & \\ \dfrac{\partial a_2}{\partial s_1} & \dfrac{\partial a_2}{\partial s_2} \end{bmatrix} = \begin{bmatrix} -he^{s_1 h} & -he^{s_2 h} \\ & \\ he^{(s_1+s_2)h} & he^{(s_1+s_2)h} \end{bmatrix} =$$

$$= h \begin{bmatrix} -e^{(-\alpha+\delta)h} & -e^{(-\alpha-\delta)h} \\ & \\ e^{-2\alpha h} & e^{-2\alpha h} \end{bmatrix} \qquad (3.1\text{-}122)$$

where the relations (3.1-117) have been used and the quantities

$$\alpha = -\frac{s_1 + s_2}{2} \tag{3.1-123}$$

and

$$\delta = \frac{s_1 - s_2}{2} \tag{3.1-124}$$

are introduced.

The error occurring in the poles of the continuous system can be reduced to an optimal extent by minimizing any scalar measure of the sensitivity matrix $\underline{\underline{E}}$ according to h. The scalar measure of the insensitivity matrix $\underline{\underline{Q}}$ has to be maximized. (The task is therefore not the extremizing of the function with scalar value but - without loss of generality - the problem can be reduced to a scalar task.)

Consider first the maximization of the determinant of $\underline{\underline{J}}(\underline{a},\underline{s})$

$$\left| \det \left[\underline{\underline{J}}(\underline{a},\underline{s}) \right] \right| = Q_1 = h^2 \left[e^{-2\alpha h} \left(e^{(-\alpha+\delta)\,h} - e^{(-\alpha-\delta)\,h} \right) \right] =$$

$$= 2h^2 e^{-3\alpha h}\, \mathrm{sh}\,\delta h \ . \tag{3.1-125}$$

The maximum of this cost function is obtained from the equation

$$\frac{\partial Q_1}{\partial h} = 0 = 4h\, e^{-3\alpha h}\, \mathrm{sh}\,\delta h - 6\alpha h^2 e^{-3\alpha h}\, \mathrm{sh}\,\delta h + 2\delta h^2\, e^{-3\alpha h}\, \mathrm{ch}\,\delta h.$$

Hence by rearrangement the nonlinear equation

$$\text{th}\,\delta h = -\frac{\delta h}{2-3\alpha h} = \frac{\delta h}{3\alpha h-2} = \frac{1}{2}\mu + \frac{1}{2}\mu^2 \frac{1}{\delta h-\mu} \qquad (3.1\text{-}126)$$

is obtained whose solution for h yields the optimal sampling period. Here the quantity

$$\mu = \frac{2\delta}{3\alpha} \qquad (3.1\text{-}127)$$

has been introduced.

Simple considerations enable us to delimit the optimal solution. The function th is always less than 1. So the Eq. (3.1-126) can be transformed to the inequality

$$1 > -\frac{\delta h}{2-3\alpha h}\,.$$

Hence the optimal sampling time

$$h_{opt} < \frac{2}{3\alpha-\delta} = \frac{2}{2\alpha+(\alpha-\delta)}\,.$$

Because of the stability the condition $\delta \leq |\alpha|$ has to be fulfilled, at the limit of the stability

$$h_{opt} < \frac{1}{\alpha}\,.$$

The nonlinear equation to be solved can be rearranged also for the relative sampling rate used up to now $x = \alpha \cdot h$:

$$x - \frac{2}{3} = \frac{\frac{\delta}{\alpha} x}{\operatorname{th} \frac{\delta}{\alpha} x} = \frac{\frac{\sqrt{\xi^2 - 1}}{\xi} x}{\operatorname{th} \frac{\sqrt{\xi^2 - 1}}{\xi} x} \qquad (3.1\text{-}128)$$

where the simply understandable relation

$$\frac{\delta}{\alpha} = \frac{s_1 - s_2}{s_1 + s_2} = \frac{\sqrt{\xi^2 - 1}}{\xi} = k(\xi) \qquad (3.1\text{-}129)$$

was taken into account. The nonlinear equation, although it can be reduced to the simpler form

$$e^{2kx} = \frac{(1-k)x - 2/3}{(1+k)x - 2/3}$$

by identical transformations, can finally be solved for a given ξ only numerically, not analytically. Thus a function relation $x_{opt}(\xi)$ could be defined by a numerical method.

Consider the solution of Eq. (3.1-126) for some special cases.

In case of identical poles $\det\left[\underline{\underline{J}}(\underline{a}, \underline{s})\right] = 0$. Thus in case of identical poles, the optimal sampling can be determined by the assumption of $\delta h \ll 1$. The estimation of the function th around zero is:

$$\operatorname{th}\delta h = \frac{e^{\delta h} - e^{-\delta h}}{e^{\delta h} + e^{-\delta h}} \approx \delta h, \qquad \delta h \ll 1.$$

Thus from Eq. (3.1-126)

$$\delta h \approx - \frac{\delta h}{2 - 3\alpha h} = \frac{\delta h}{3\alpha h - 2}$$

whence

$$h_{opt} \approx \frac{1}{\alpha} . \qquad (3.1\text{-}130)$$

In the case of imaginary poles (damped oscillatory system), the substitution $\delta = j\delta$ have to be applied and the equation

$$\mathrm{tg}\ \delta h = - \frac{\delta h}{2-3\alpha h} = \frac{\delta h}{3\alpha h-2} \qquad (3.1\text{-}131)$$

is obtained.

Please find here below some hints for the solution of the nonlinear equation:

if $\mu < \frac{\pi}{2}$ then $\delta h_{opt} < \frac{\pi}{2}$

if $\mu = \frac{\pi}{2} \longrightarrow h_{opt} = \frac{2}{3\alpha}$, and thus $\delta h_{opt} = \frac{\pi}{2}$

if $\P > \mu > \frac{\pi}{2}$ then $\delta h_{opt} > \frac{\pi}{2}$

$$(3.1\text{-}132)$$

In case of the undamped oscillatory system $\alpha = 0$, such that the equation (3.1-131) will take the form

$$\operatorname{tg} \delta h = - \frac{\delta h}{2} \tag{3.1-133}$$

whence

$$\delta h_{opt} \approx 1{,}142$$

i.e.

$$\delta h_{opt} = \omega h_{opt} = \frac{2\pi}{T} h_{opt} = 1{,}142$$

that is

$$\frac{h_{opt}}{T} = \frac{1{,}14}{2\pi} = 0{,}1818 \ .$$

Therefore about three samples are to be taken from a sine wave. (A simple calculation shows that in this case $\phi \approx 131^o$, so that no complete left half-circle, only a part of the same is required.)

As according to Eq. (3.1-121), $\underline{\underline{Q}}$ is the product of two matrices, but $\underline{\underline{S}}$ does not depend on h, such that the maximization of the determinant of $\underline{\underline{J}}(\underline{a},\underline{s})$ has maximized the determinant of $\underline{\underline{Q}}$, too.

Consider now the maximization of the trace of the Jacobian matrix in the following equation:

$$\begin{bmatrix} \Delta a_1 \\ \\ \Delta a_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial a_1}{\partial \alpha} & \dfrac{\partial a_1}{\partial \delta} \\ \\ \dfrac{\partial a_2}{\partial \alpha} & \dfrac{\partial a_2}{\partial \delta} \end{bmatrix} \begin{bmatrix} \Delta \alpha \\ \\ \Delta \delta \end{bmatrix} =$$

$$= \begin{bmatrix} -h\, e^{-\alpha h} \left(e^{\delta h} + e^{-\delta h}\right) & - he^{-\alpha h} \left(e^{\delta h} - e^{-\delta h}\right) \\ \\ 2\, he^{-2\alpha h} & 0 \end{bmatrix} . \qquad (3.1\text{-}134)$$

This task ensures the insensitivity of the location of the poles of the continuous system by maximizing the trace through the quantities α and δ. The maximum of the trace of the matrix

$$Q_2 = \mathrm{tr}\left[\underline{\underline{J}}(\underline{a}, \alpha\, \delta)\right] = 2he^{-2\alpha h}\, \mathrm{th}\, \delta h \qquad (3.1\text{-}135)$$

is obtained from the equation

$$\frac{\partial Q_2}{\partial h} = 0 = \left[(1 - 2\alpha h)\ \mathrm{th}\, \delta h + 2\, \delta h \frac{1}{\mathrm{ch}^2\ \delta h}\right] e^{-2\alpha h} =$$

$$= \frac{e^{-2\alpha h}}{\mathrm{ch}^2 \delta h} \left[(1 - 2\alpha h)\ \mathrm{sh}\, (2\, \delta\, h) + 2\delta h\right] .$$

As the first term is not equal to zero, thus

$$(1 - 2\alpha h)\ \mathrm{sh}\ (2\, \delta\, h) + 2\delta h = 0$$

whence the nonlinear equation

$$\operatorname{sh} 2\delta h = - \frac{2\delta h}{1 - 2\alpha h} = \frac{2\delta h}{2\alpha h - 1} \tag{3.1-136}$$

is obtained whose solution ensures the optimal sampling period.

As $\operatorname{sh} \mathbf{x} > 0$, if $\mathbf{x} > 0$, then by applying these conditions to the Eq. (3.1-136)

$$1 - 2\alpha h < 0$$

i.e.

$$h_{opt} > \frac{1}{2\alpha}$$

and using this in (3.1-136), we get to the inequality

$$\frac{1}{2\alpha} < h_{opt} < \frac{1}{\alpha}$$

for the delimitation of the optimal sampling period.

When investigating complex poles the substitution $\delta = j\delta$ has to be applied. Then instead of Eq. (3.1-135), we have to determine the maximum of the quantity

$$Q_2 = 2h\, e^{-2\alpha h} \operatorname{tg} \delta h \quad . \tag{3.1-137}$$

It is easy to see that it takes its maximum at the place $\delta h_{opt} = \frac{\pi}{2}$. The optimal sampling period corresponding to this is

$$h_{opt} = \frac{1}{2\alpha} .$$

(Note that the equivalence of the determinants of the matrices in (3.1-134) and (3.1-122) is easy to be seen. The maximization of the trace of the matrix $\underline{\underline{J}}(\underline{a},\underline{s})$ can be obtained through the solution of the nonlinear equation

$$(\alpha-\delta)\, h = \ln \frac{1-2\,\alpha\, h}{1-(\alpha+\delta)h}).$$

In this subsection the sensitivity of the pole transformation have been investigated by seeking the possibility of the optimization of the sampling period with the purpose to reduce as much as possible the occurrence of the uncertainties, arising at the identification in the poles or parameters of the discrete system, in the poles of the continuous system obtained by a transformation giving equivalence for the step input signal. That sampling period was considered optimal which ensured the maximal insensitivity of the poles to changes caused by the estimation.

The most important experiences can be summarized as follows. The optimal sampling time is strongly criterion-dependent and it is very difficult to tell which is the best criterion. The first relations obtained for first and second-order systems seem to be the most useful and by advancing toward more complicated criteria we arrived at equations whose solution and interpretation became even more difficult. Results supported by diagrams are suggested to determine the optimal sampling time and to investigate the optimal location of the poles of the discrete forms obtained by identification.

The most important experience of sampling period optimization on the basis of the aforesaid was that not the shortest possible sampling time can be considered as best for identification but a value coinciding with a time constant corresponding to the medium-frequency domain of the system.

3.2 Minimization of the sensitivity of zero transformation

Consider first a first-order system when using step response equivalent transformation the continuous transfer function $H(s)=\beta_1/(s+\alpha_1)$ and the discrete transfer function $G(z)= b_1/(z+a_1)$ are compared, cf. Chapters 2 and 3.1. In this case the transfer functions have no zeroes, it is practical to investigate the sensitivity relation of the quantities (parameters) β_1 and b_1. These quantities are the numerical values of the residues belonging to the corresponding poles. Define now the following sensitivity function

$$E \hat{=} \frac{\left|\frac{\partial \beta_1}{\partial b_1}\right|}{\beta_1} \qquad (3.2\text{-}1)$$

i.e. the corresponding insensitivity function

$$Q = \frac{1}{E} = \beta_1 \frac{1}{\left|\frac{\partial \beta_1}{\partial b_1}\right|} \quad . \qquad (3.2\text{-}2)$$

The derivative figuring here is on the basis of relations (3.1-1)

$$\frac{\partial \beta_1}{\partial b_1} = \frac{\partial}{\partial b_1}\left(\frac{b_1 \alpha_1}{1+a_1}\right) = \frac{\alpha_1}{1+a_1} = \frac{\alpha_1}{1-e^{-h\alpha_1}} = \frac{\alpha_1}{1-e^{-x}} \; . \qquad (3.2\text{-}3)$$

Thus the insensitivity function is

$$Q = \beta_1 \frac{1-e^{-x}}{\alpha_1} = K(1-e^{-x}) \; . \qquad (3.2\text{-}4)$$

The function apparently has its maximum at the place $x = \infty$.

Consider now the effect of the uncertainty of the discrete pole on the residue of the continuous system. Practically,

$$Q \triangleq \beta_1 \frac{1}{\left| \dfrac{\partial \beta_1}{\partial p_1} \right|}$$

where $p_1 = -a_1$ is the pole of the discrete system. By using the relation

$$\frac{\partial \beta_1}{\partial p_1} = \frac{\partial}{\partial p_1} \frac{b_1 \alpha_1}{(1+a_1)} = \frac{\partial}{\partial p_1} \frac{b_1 \alpha_1}{1 - p_1} = \frac{b_1 \alpha_1}{(1-p_1)^2}$$

the insensitivity function is

$$Q(x) = \beta_1 \frac{(1 - p_1)^2}{b_1 \alpha_1} = \beta_1 \frac{1 + a_1}{b_1 \alpha_1} (1-e^{-x}) = 1-e^{-x} \longrightarrow \max_x .$$

This again means that the infinitely long sampling period can be considered as optimal. As β_1 can be considered gain factor it is also physically understandable that its value can be most exactly estimated on the basis of the steady state $(x=\infty)$. As the gain factor K is affected by the uncertainties of both the pole and the zero (now b_1), therefore, the development of the product insensitivity

$$Q(x) = \frac{K}{\left| \dfrac{\partial K}{\partial p_1} \right|} \cdot \frac{K}{\left| \dfrac{\partial K}{\partial b_1} \right|} = K^2 \left[\frac{\partial}{\partial p_1} \frac{b_1}{1-p_1} \right]^{-1} \left[\frac{\partial}{\partial b_1} \frac{b_1}{1-p_1} \right]^{-1} =$$

$$= K^2 \frac{(1+a_1)^2}{b_1} (1-e^{-x}) = K(1-e^{-x})^2 \longrightarrow \max_x$$

is essential. It is easy to see that the result $x_{opt} = \infty$, have been obtained again.

Let now the form of the transfer function of the continuous system be

$$H(s) = \frac{\beta_1 + \beta_0 s}{\alpha_1 + s} = K \frac{1 + T_1 s}{1 + Ts} . \qquad (3.2\text{-}5)$$

By a step response equivalent transformation the discrete transfer function

$$G(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}} = \frac{(b_1 - b_0 a_1)\, z^{-1}}{1 + a_1 z^{-1}} + b_0 \qquad (3.2\text{-}6)$$

can be made to correspond to this continuous system. On the basis of (2.16) and (2.17), the transformational relations are the followings:

$$\alpha_1 = - \frac{\ln(-a_1)}{h} ; \quad \beta_1 = \frac{b_0 + b_1}{1 + a_1} \alpha_1 ; \quad \beta_0 = b_0 \qquad (3.2\text{-}7)$$

respectively

$$K = \frac{b_0 + b_1}{1 + a_1} ; \quad T = - \frac{h}{\ln(-a_1)} ; \quad T_1 = \frac{b_0}{K} T . \qquad (3.2\text{-}8)$$

The zero of the continuous system is

$$s_1 = - \frac{\beta_1}{\beta_0} = - \frac{1}{T_1} . \qquad (3.2\text{-}9)$$

That of the discrete system

$$z_1 = - \frac{b_1}{b_o} . \tag{3.2-10}$$

Convert the latter expression

$$z_1 = - \frac{b_1}{b_o} = - \frac{1}{b_o} \left[\frac{(1 + a_1) \beta_1}{\alpha_1} - b_o \right] =$$

$$=1- \frac{(1-e^{-x}) \beta_1}{\alpha_1 \beta_o} = 1 - \frac{K}{b_o} (1 - e^{-x}) = 1- \frac{T}{T_1} (1-e^{-x}), \tag{3.2-11}$$

where we have considered that $a_1 = -e^{-x}$ and $x = h/T$. Note that in the obtained relation the

$$\frac{T}{T_1} = \frac{\text{zero}}{\text{pole}} = \left(- \frac{1}{T_1}\right) : \left(- \frac{1}{T}\right) \tag{3.2-12}$$

coincides with the zero-pole relation. Hence the zero transformation is realized, according to Fig. 3.2-1, viz.

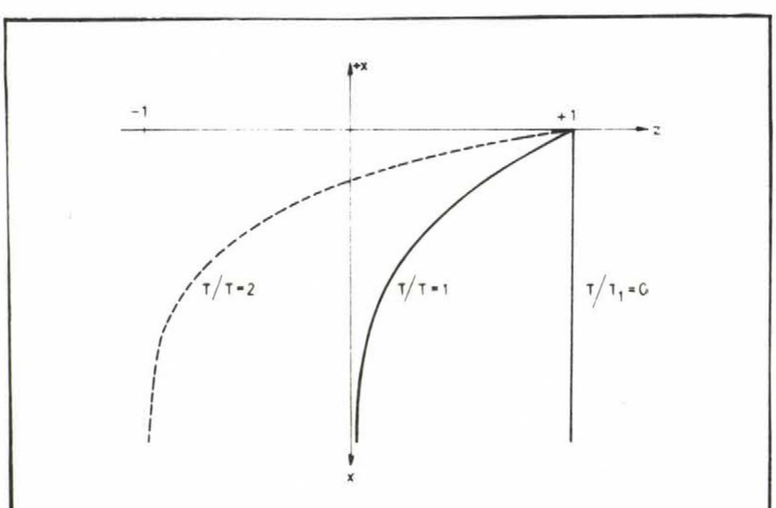

Fig. 3.2-1

T/T_1	0	1	2	...
$z_1\ (x=\infty) = -\frac{T}{T_1}+1$	1	0	-1	...

(Note that as a matter of curiosity, that the condition $z_1=0$ can be attained by the sampling time $x = -\ln\frac{(T-T_1)}{T}$).

Consider the sensitivity of the transformation.

Express to this the continuous zero s_1 by the discrete zero z_1

$$s_1 = -\frac{\beta_1}{\beta_o} = \frac{\frac{b_o+b_1}{1+a_1}\alpha_1}{b_o} = \frac{(b_o+b_1)\ \ln(-a_1)}{(1+a_1)\ h\ b_o} =$$

$$=\left(1+\frac{b_1}{b_o}\right)\frac{\ln(-a_1)}{h(1+a_1)} = \frac{(1-z_1)\ \ln(-a_1)}{h(1+a_1)} =$$

$$= -\frac{(1-z_1)\ x}{h\ (1-e^{-x})} = -\frac{1-z_1}{T\ (1-e^{-x})}\ . \qquad (3.2\text{-}13)$$

Form now the insensitivity function

$$Q(x) = \frac{|s_1|}{\left|\frac{\partial s_1}{\partial z_1}\right|} = \frac{1}{T_1\left|\frac{\partial s_1}{\partial z_1}\right|} \qquad (3.2\text{-}14)$$

where on the basis of (3.2-13)

$$\frac{\partial s_1}{\partial z_1} = \frac{1}{T} \frac{1}{1 - e^{-x}} , \tag{3.2-15}$$

and thus

$$Q(x) = \frac{1}{T_1} \left[\frac{1}{T} \frac{1}{1-e^{-x}} \right]^{-1} = \frac{T}{T_1} (1-e^{-x}) = \frac{K}{\beta_o} (1-e^{-x}). \tag{3.2-16}$$

Fig. 3.2-2 shows the function $Q(x)$ according to which the case $x_{opt} = \infty$ ensures maximal insensitivity.

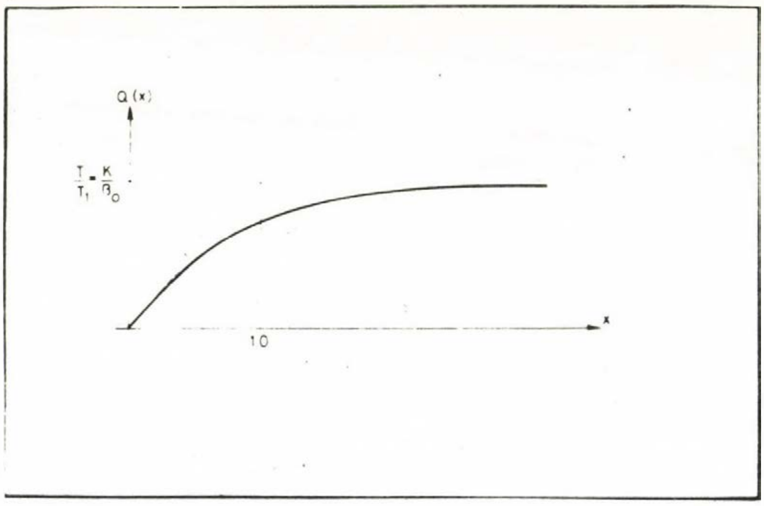

Fig. 3.2-2

Express now the continuous zero s_1 by the pole p_1 of the discrete system. As $p_1 = -a_1 = e^{-x}$, so

$$s_1 = \frac{(1 + \frac{b_1}{b_o}) \ln(-a_1)}{h\ (1+a_1)} = \frac{(1 - z_1) \ln p_1}{h(1-p_1)} . \tag{3.2-17}$$

Considering that (cf. Eq. (3.2-11))

$$1 - z_1 = \frac{T}{T_1} (1-e^{-x}) = \frac{T}{T_1} (1+a_1) = \frac{T}{T_1} (1-p_1) , \tag{3.2-18}$$

the following relation is obtained:

$$s_1 = \frac{\frac{T}{T_1}(1-p_1)\ \ln p_1}{h(1-p_1)} = \frac{T}{T_1 h} \ln p_1 = \frac{1}{T_1 x} \ln p_1 . \qquad (3.2\text{-}19)$$

Hence

$$\frac{\partial s_1}{\partial p_1} = \frac{1}{T_1 x}\ \frac{1}{p_1} = \frac{1}{T_1 x}\ \frac{1}{e^{-x}} = \frac{1}{T_1}\ \frac{e^x}{x} \qquad (3.2\text{-}20)$$

that is the corresponding insensitivity function

$$Q(x) = \frac{1}{T_1 \left|\frac{\partial s_1}{\partial p_1}\right|} = \frac{1}{T_1 \frac{e^x}{T_1 x}} = x\ e^{-x}. \qquad (3.2\text{-}21)$$

Such that this function coincides with the reciprocal of $E(x)$ according to Eq. (3.1-8), so that the optimal relative sampling rate is now $x_{opt} = 1$. Therefore, the uncertainty of the discrete pole causes in the case $x=1$ both to the continuous pole and to the continuous zero the least trouble.

Examine now, too, the insensitivity function of K in the form of the product:

$$Q(x) = \frac{K}{\left|\frac{\partial K}{\partial z_1}\right|} \cdot \frac{K}{\left|\frac{\partial K}{\partial p_1}\right|} =$$

$$= K^2 \left[\frac{\partial}{\partial z_1} K \frac{T_1}{T} \frac{(1-z_1)}{(1-p_1)}\right]^{-1} \cdot \left[\frac{\partial}{\partial p_1} K \frac{T_1}{T} \frac{(1-z_1)}{(1-p_1)}\right]^{-1} =$$

$$= K^2 \left[\frac{KT_1}{T}(-1)(1-p_1)\right]^{-1} \left[\frac{KT_1}{T}(-1)(1-z_1)(-1)\frac{1}{(1-p_1)^2}\right]^{-1} =$$

$$= \frac{T^2}{T_1\ (T-T_1)}\ (1-e^{-x}) \longrightarrow \max_{x} .$$

We came this way again to the conclusion that the maximal insensitivity of the gain factor K can be ensured by an infinitely long sampling period (steady state).

Observe now that examining the pole transformation the pole of the continuous system was disturbed only by the uncertainty of the pole of the discrete system, while the coefficients in the numerator - be it residue, zero or gain factor - are sensitive to the uncertainties of both discrete pole and zero. Therefore we produce from the two kinds of insensitivities again a product insensitivity in the form of

$$Q(x) = \frac{1}{T_1 \left|\frac{\partial s_1}{\partial z_1}\right|} \cdot \frac{1}{T_1 \left|\frac{\partial s_1}{\partial p_1}\right|} . \qquad (3.2\text{-}22)$$

On the basis of the relations (3.2-16) and (3.2-21), we can write that

$$Q(x) = \frac{T}{T_1}\ (1-e^{-x})\ x\ e^{-x} = \frac{T}{T_1}\ \frac{x(1-e^{-x})}{e^{x}} . \qquad (3.2\text{-}23)$$

In order to determine the extremum value we obtain from the condition

$$\frac{\partial Q(x)}{\partial x} = \frac{T}{T_1} \frac{\left[(1-e^{-x}) + x\, e^{-x}\right] e^{x} - x(1-e^{-x})\, e^{x}}{e^{2x}} =$$

$$= \frac{T}{T_1\, e^{x}} \left[(1-x) - (1-2x)\, e^{-x}\right] = 0 \qquad (3.2\text{-}24)$$

the nonlinear equation

$$e^{x} = \frac{2x-1}{x-1} \qquad (3.2\text{-}25)$$

whose solution yields the optimal sampling rate for the domain $x > 0$. Fig. 3.2-3 represents the left and right sides

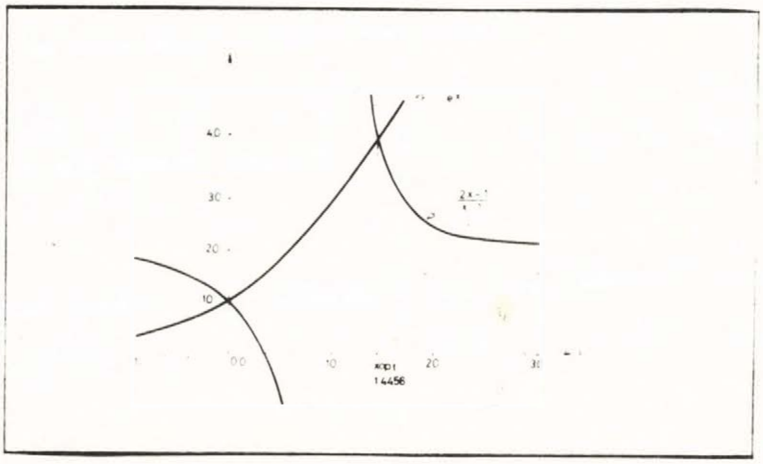

Fig. 3.2-3

of the equation, that is the inequality $1.0 < x_{opt} < 2.0$ must hold. The numerical solution yields as a result the optimal relative sampling rate

$$x_{opt} = 1.4456 \ . \qquad (3.2\text{-}26)$$

(There is an inflexion point in the place x=0). Thus the optimal sampling period with the zero transformation is somewhat longer than that with the pole transformation.

For curiosity examine now the insensitivity as a sum according to

$$Q(x) = \frac{1}{T_1 \left|\frac{\partial s_1}{\partial z_1}\right|} + \frac{1}{T_1 \left|\frac{\partial s_1}{\partial p_1}\right|} = \frac{T}{T_1}\,(1-e^{-x}) + xe^{-x}. \qquad (3.2\text{-}27)$$

In order to determine the extremum value we obtain from the condition

$$\frac{\partial Q(x)}{\partial x} = \frac{T}{T_1}\; e^{-x} + e^{-x} + x(-e^{-x}) = \left(\frac{T}{T_1} + 1-x\right)\; e^{-x} = 0 \qquad (3.2\text{-}28)$$

the optimal sampling rate

$$x_{opt} = 1 + \frac{T}{T_1}\;. \qquad (3.2\text{-}29)$$

This relation means that if T is substantially greater than T_1, then according to what was said for the static gain, a sampling period, as long as possible, would be needed, if, on the other hand, it is substantially less, then the same rule is valid for the zero as for the pole.

In the relative insensitivities it would have been perhaps more appropriate to use instead of $\left|-\frac{1}{T_1}\right|$ the quantity $\left|-\frac{1}{T}\right|$, for the case $T \longrightarrow 0$ surely will not occur, while $T_1 \longrightarrow 0$ will. It is easy to understand that our statements relating to the optimal sampling time would continue to hold invariably.

The relations covering the zero of the second-order system are very complicated, therefore they will not be discussed here.

3.3 Joint sensitivity tests

The insensitivity (sensitivity) matrix techniques discussed in Chapter 3.1, would best suit for the joint sensitivity tests. The product sensitivities (although widespread in the sensitivity analysis via the logarithmic forms) are not the most informative in case of several parameters, in case of interactions, they do not have meaning. For analytical tests, on the other hand, only 2x2 matrices can be simply used, at most, such that in case of parameter more than these we have to turn to product (or possibly sum) sensitivity functions.

The following insensitivity matrix can be defined for the first-order system (3.2-5)

$$\underline{\underline{Q}} = \begin{bmatrix} \frac{1}{T_1} \left(\frac{\partial Z_1}{\partial z_1}\right)^{-1} & \frac{1}{T_1} \left(\frac{\partial Z_1}{\partial p_1}\right)^{-1} \\ & \\ \frac{1}{T} \left(\frac{\partial P_1}{\partial z_1}\right)^{-1} & \frac{1}{T} \left(\frac{\partial P_1}{\partial p_1}\right)^{-1} \end{bmatrix} = \begin{bmatrix} \frac{T}{T_1}(1-e^{-x}) & xe^{-x} \\ & \\ 0 & xe^{-x} \end{bmatrix} \tag{3.3-1}$$

where Z_1 and P_1 are the zero and pole of the continuous, z_1 and p_1 of the discrete system, resp. (These notations were not needed heretofore but present joint tests have made them necessary to ensure unambiguity.) On the basis of (3.2-16), (3.2-21) and (3.1-8), we have set the particular elements into the insensitivity matrix (3.3-1). On the basis of the relation, it can be established that the maximization of the determinant $Q_1 = \det(\underline{\underline{Q}})$ can be obtained by the maximization of the product sensitivity (3.2-22). The optimal sampling period is determined therefore by (3.2-26). The maximization of the trace $Q_2 = tr(\underline{Q})$ involves the maximization of the sum insensitivity (3.2-27) and the optimal sampling period will be governed by (3.2-29) also in this case.

It has been mentioned in the preceding Chapter that it is more reasonable to relate the changes to the quantity $1/T$. The insensitivity matrix will then be as follows:

$$\underline{Q} = \begin{bmatrix} \frac{1}{T}\left(\frac{\partial Z_1}{\partial z_1}\right)^{-1} & \frac{1}{T}\left(\frac{\partial Z_1}{\partial p_1}\right)^{-1} \\ \\ \frac{1}{T}\left(\frac{\partial P_1}{\partial z_1}\right)^{-1} & \frac{1}{T}\left(\frac{\partial P_1}{\partial p_1}\right)^{-1} \end{bmatrix} = \begin{bmatrix} 1 - e^{-x} & xe^{-x} \\ \\ 0 & xe^{-x} \end{bmatrix} . \tag{3.3-2}$$

The place of the maximum of the determinant accordingly does not change, but the maximum of $tr(\underline{Q})$ does, for now

$$Q_2 = tr(Q) = 1 - e^{-x} + xe^{-x} . \tag{3.3-3}$$

The maximum of the above quantity can be obtained by ensuring the condition

$$\frac{\partial Q_2(x)}{\partial x} = e^{-x} + e^{-x} - xe^{-x} = (2 - x)\, e^{-x} = 0 . \tag{3.3-4}$$

Hence the optimal sampling rate is

$$x_{opt} = 2 . \tag{3.3-5}$$

Accordingly the optimal discrete pole and zero are

$$p_1 = -a_1 = e^{-x} = e^{-2} = 0.1353 \qquad (3.3\text{-}6)$$

and

$$z_1 = 1 - \frac{T}{T_1}\,(1-e^{-2}) = 1 - 0.8647\,\frac{T}{T_1} \quad . \qquad (3.3\text{-}7)$$

Experiences show therefore that if in the numerator of the continuous system, the static transfer prevails, then a sampling period as long as possible is desirable. If the zero indicating the dynamic effect dominates, then the optimal sampling periods shifts towards the optimal medium-frequency domain resulting from the investigation of the denominator of the continuous system. The sensitivity test of the denominator of the continuous system has ensured an optimal sampling period corresponding to a medium-frequency domain dependent on the chosen optimality criterion. The joint tests have indicated that the joint sensitivity of the numerator and denominator requires some sort of compromise between the above two conditions. So that the most important statement is that, not the striving for the shortest possible sampling time involves optimality.

The joint sensitivity test for higher-order systems leads to very complicated calculations and enables numerical evaluation only. Although, neither this approach should be underestimated from the viewpoint of checking identification tests, only the analytic results yielded a solution which could be easily generalized.

On the basis of the relations (2.50), (2.53) and (2.77), we can write

$$\underline{g} = (e^{\underline{\underline{A}}h} - \underline{\underline{I}})\, \underline{\underline{A}}^{-1}\, \underline{\underline{T}}^{-1}\, \underline{\underline{P}}^{-1}\, \underline{q} \qquad (3.3\text{-}8)$$

which apparently refers to the proportional growth with h of the insensitivity of the parameters in the numerator.

In the relationship of the parameters in the denominator, we draw the attention to the very close formal analogy between the equation

$$\underline{k}^T\, \underline{\underline{T}} = -\frac{1}{h^n}\, \underline{c}^T\, \left[\ln(\underline{\underline{F}})\right]^n \qquad (3.3\text{-}9)$$

which can be obtained from the relations (2.50) and (2.75), and the transformation given in (3.1-6). It is easy to see that if $\underline{\underline{F}}$ would not contain the poles of the discrete system in a matrix form, then the insensitivity test could be carried out very easily. Thus we point only to the expectation of the denotation of a medium-frequency domain alike the function xe^{-x}, when determining the optimal sampling rate.

IV. APPROACHES BASED ON ESTIMATION THEORY

Parameter estimation is an important step of process identification. E.g. the up-to-date discrete identification methods give the parameter estimation of the discrete models equivalent in the sampling instants.

Following from the character of the estimation, the determination of these parameters has a statistical uncertainty. This uncertainty extends through the discrete-continuous equivalent transformation also to the determination, the estimation of the parameters of the continuous system. In the preceding chapters only the sensitivity of the discrete-continuous transformation was tested when we discussed the optimization of the sampling period. We are going now to deal with optimization possibilities resulting from the estimation task.

Let the theoretical parameter vector be $\underline{p}$ and its estimated value denoted by $\hat{\underline{p}}$. Let us name

$$\underline{\underline{K}} = E\{(\hat{\underline{p}} - \underline{p})(\hat{\underline{p}} - \underline{p})^T\} \tag{4-1}$$

the covariance matrix of the estimated parameters, where $E\{\ldots\}$ denotes the expected value. (Here and later unbiased or asymptotically unbiased estimation is dealt with.) As it is known [1], [27], the covariance matrix contains the quantities characteristic for the uncertainty, accuracy (variances in the main diagonal) of the estimated parameter vector, as well as of the interaction of the estimates (covariances out of the main diagonal).

As to the information obtainable about the process, the

CRAMER-RAO theorem is of special importance according to which by processing N number of samples with a parameter estimation method, the inequality

$$\underline{\underline{K}}_N \geq \underline{\underline{J}}_N^{-1} \tag{4-2}$$

gives the lower bound from the point of view of information and estimation theory. Here $\underline{\underline{J}}_N$ is the FISHER information matrix relating to N samples [5], [7]. The inequality (4-2) means - out of several kinds of equivalent interpretations - that the difference matrix $\underline{\underline{K}}_N - \underline{\underline{J}}_N^{-1}$ is not a negative definite, i.e. the variance of the i-th element of the estimated parameter vector $\hat{\underline{p}}$ can not be less than the i-th diagonal element of $\underline{\underline{J}}_N^{-1}$.

According to the definition of the information matrix

$$\underline{\underline{J}} = E\{\underline{g}\ \underline{g}^T\} \tag{4-3}$$

where

$$\underline{g} = \frac{d\ L(\underline{p})}{d(\underline{p})} \tag{4-4}$$

is the gradient of the likelihood function by the parameters [59], [63]. The likelihood function $L(\underline{p})$ is the logarithm of the conditional probability density function $d(\underline{e}|\underline{p})$ of the estimation error vector $\underline{e}$ used for the construction of the parameter estimation method. Hence

$$L(\underline{p}) = \ln d(\underline{e}|\underline{p})\ . \tag{4-5}$$

So that the definition (4-3) can be set also this way:

$$\underline{\underline{J}} = \int_{\Omega} d(\underline{e}|\underline{p}) \left[\frac{d}{d\underline{p}} \ln d(\underline{e}|\underline{p})\right] \left[\frac{d}{d\underline{p}} \ln d(\underline{e}|\underline{p})\right]^T d\underline{e} \tag{4-6}$$

where Ω denotes the complete space which $\underline{e}$ can take up.

With practical tests instead of the real error vector $\underline{e}$ the residual vector $\underline{\varepsilon}$ resulting from the estimation is used to be applied.

The attainment, the approach of the lower CRAMER-RAO bound is virtually method-dependent. Accordingly, when optimizing now the sampling period, we are optimizing the qualitative characteristics of the theoretically obtainable best estimation resulting from the information matrix and not engaging in the search of the best estimation method. Our objective by this approach is the maximization of the information which can be obtained about the process and not the optimization of a given method (although we shall revert to this latter case).

The notion of the information matrix, as in general the other methods connected with the maximum likelihood methods, leads in case of normally distributed error to relations easy to use. As the whole discrete identification technique takes advantage of this assumption, it is advisable also to us to assume hereinafter the same. In our system equations the measurement error (ε_k) appears additively, so that the measured output signal (y_k) is the sum of the noiseless output signal v_k and the measurement noise, i.e.

$$y_k = v_k + \varepsilon_k . \tag{4-7}$$

Assuming the normal distribution for the noise with zero mean ($E\{\varepsilon(t)\} = 0$), its probability density function will take the form

$$d(\underline{\varepsilon}|\underline{p}) = k_1 \exp\{-\tfrac{1}{2}(\underline{y} - \underline{v}) \; \underline{\underline{W}}^{-1} \; (\underline{y} - \underline{v})\} =$$

$$= k_1 \exp\{-\tfrac{1}{2} \underline{\varepsilon} \; \underline{\underline{W}}^{-1} \; \underline{\varepsilon}\} \tag{4-8}$$

where $\underline{y}$, $\underline{v}$ and $\underline{\varepsilon}$ refer to the vector form of Eq. (4-7)

$$\underline{y} = \underline{v} + \underline{\varepsilon} \tag{4-9}$$

valid for N samples, moreover

$$E\{\underline{\varepsilon} \; \underline{\varepsilon}^T\} = \underline{\underline{W}} \tag{4-10}$$

is the covariance matrix of the noise. Thus, the likelihood function

$$L(\underline{p}) = k_2 - \tfrac{1}{2}(\underline{y} - \underline{v})^T \underline{\underline{W}}^{-1}(\underline{y} - \underline{v}) = k_2 - \tfrac{1}{2} \underline{\varepsilon}^T \underline{\underline{W}}^{-1} \underline{\varepsilon} \tag{4-11}$$

where the noiseless output of the system $\underline{v} = \underline{v}(\underline{p})$ is parameter-dependent and $\underline{y}$ contains the measured values. Hence

$$\underline{g} = \frac{d\,L(\underline{p})}{d\,\underline{p}} = \frac{d\,\underline{v}^T}{d\,\underline{p}} \; \frac{d\,L(\underline{p})}{d\underline{v}} = \frac{d\,\underline{v}^T}{d\,\underline{p}} \; \underline{\underline{W}}^{-1}(\underline{y} - \underline{v}) = \frac{d\,\underline{v}^T}{d\,\underline{p}} \underline{\underline{W}}^{-1} \underline{\varepsilon}.$$

$$\tag{4-12}$$

According to the definition (4-6)

$$\underline{\underline{J}} = \int_{\Omega} d(\underline{\varepsilon}|\underline{p}) \left[\frac{d\,\underline{v}^T}{d\,\underline{p}} \underline{\underline{W}}^{-1} \underline{\varepsilon} \right] \left[\frac{d\,\underline{v}^T}{d\,\underline{p}} \underline{\underline{W}}^{-1} \underline{\varepsilon} \right]^T d\underline{\varepsilon} =$$

$$= \frac{d\,\underline{v}^T}{d\,\underline{p}} \underline{\underline{W}}^{-1} E\{\underline{\varepsilon}\,\underline{\varepsilon}^T\} \underline{\underline{W}}^{-1} \frac{d\,\underline{v}}{d\,\underline{p}^T} = \frac{d\,\underline{v}^T}{d\,\underline{p}} \underline{\underline{W}}^{-1} \frac{d\,\underline{v}}{d\,\underline{p}^T} \qquad (4\text{-}13)$$

where the relation (4-10) and the fact that $\underline{\underline{W}}$ is symmetric were used. Inasmuch, the structure of $\underline{\underline{W}}$ is of a special kind (i.e. is related to a uniformly distributed, uncorrelated noise)

$$\underline{\underline{W}} = \lambda^2 \underline{\underline{I}} \qquad (4\text{-}14)$$

then also (4-13) will be simpler

$$\underline{\underline{J}} = \frac{1}{\lambda^2} \frac{d\,\underline{v}^T}{d\,\underline{p}} \frac{d\,\underline{v}}{d\,\underline{p}^T} \,. \qquad (4\text{-}15)$$

As the definition (4-6) can be written also in the form

$$\underline{\underline{J}} = E\left\{ \left[\frac{d}{d\underline{p}} L(\underline{p}) \right] \cdot \left[\frac{d}{d\underline{p}} L(\underline{p}) \right]^T \right\} \qquad (4\text{-}16)$$

this relation takes in the case of the existence of (4-14) the form [5]

$$\underline{\underline{J}} = E\left\{ \frac{1}{\lambda^2} \frac{d\,\underline{\varepsilon}^T}{d\underline{p}} \underline{\varepsilon}\,\underline{\varepsilon}^T \frac{1}{\lambda^2} \frac{d\,\underline{\varepsilon}}{d\,\underline{p}^T} \right\} =$$

$$= \frac{1}{\lambda^2} E\{ \sum_{k=1}^{N} \frac{d\varepsilon_k}{d\underline{p}} \cdot \frac{d\varepsilon_k}{d\underline{p}^T} \} = \frac{N}{\lambda^2} E\{ \frac{d\varepsilon_k}{d\underline{p}} \cdot \frac{d\varepsilon_k}{d\underline{p}^T} \} = \underline{J}_N \, . \qquad (4\text{-}17)$$

The determination of the infromation matrix requires therefore the determination of the expected value of the quantities generated from the product of the elements of $dv_k/d\underline{p}$ or of the equivalent sensitivity vector (gradient) $d\varepsilon_k/d\underline{p}$.

4.1 Models used by discrete identification methods

It was stressed in the introduction that the report would deal with single input single output linear dynamic systems where the system is assumed to be as serial connection of a term with concentrated parameters and of a term with dead-time at most. On the basis of the sampled values of the input and output signals, the discrete-time model equivalent with the original system at the sampling instants, has to be determined. In consequence the task is to perform, in the knowledge of the primary structure, the experimental identification in order to estimate the secondary structure or the parameters.

ÅSTRÖM and his co-workers have suggested in their paper cited as a source [10] the application of the model shown in Fig. 4.1-1. The model, as it can be well seen, is subdivided into two parts: an actual process model and a so-called noise

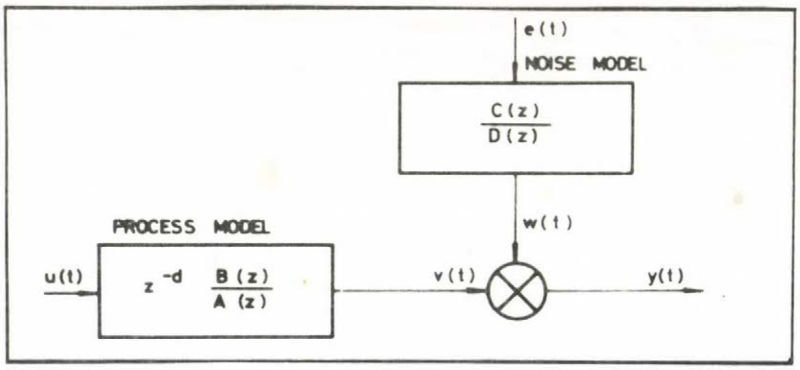

Fig. 4.1-1

model symbolizing the environmental and measurement noise. This model describes the process with the following linear stochastic difference equation:

$$y_k = \sum_{i=0}^{n} b_i\, u_{k-d-i} - \sum_{i=1}^{n} a_i\, y_{k-i} + \sum_{i=1}^{n} a_i w_{k-i} + w_k \tag{4.1-1}$$

where

$$w_k = e_k + \sum_{i=1}^{n} c_i \, e_{k-i} - \sum_{i=1}^{n} d_i w_{k-i} \, . \qquad (4.1\text{-}2)$$

u_k is here the discrete input signal of the process (cf. Chapter 2), v_k is its output signal. y_k denotes the value of the output signal measured by the additive measurement and environmental w_k noise, reduced to the output, e_k is the reason of the output noise, the so-called source noise assumed practically as a white noise with zero mean (since an output noise with arbitrary rational spectrum can be produced from it by a linear C/D filter). Let the standard deviation of e_k be λ and for ensuring the unambiguity the zero-order member of the polynomial $C(z)$ be unit. The connection of the particular signals can be followed in Fig. 4.1-1. The quantities a_i, b_i, c_i, d_i are the parameters of this system, and d is the dead-time measured in the units of the sampling interval (integer) so that its amount is dh. (Note here that because of the own delay of the inertial systems the real dead-time has the value of (d-1)h only. In applying the various parameter estimation procedures, it is a usual precondition to measure u_k without error and that its value should be uncorrelated to the source noise. For the maximum likelihood (hereinafter ML) estimation the normal distribution of e_k is a condition.)

In the control theory the use of the formal operator description in a direct way is of more reasonable (see Fig.4.1-1)

$$y_k = z^{-d} \, \frac{B(z)}{A(z)} \, u_k + \frac{C(z)}{D(z)} \, e_k = v_k + w_k \, . \qquad (4.1\text{-}3)$$

Here $B(z)/A(z)$ and $C(z)/D(z)$ are the discrete transfer functions discussed in Chapter 1.

In the relation (4.1-3) $A(z)$, $B(z)$, $C(z)$, $D(z)$ are the polynomials of z^{-1} and z^{-1} denotes the backward shift operator which virtually coincides with the inverse of the variable of the z-transformation. Therefore for an x_k signal $z^{-1}\, x_k = x_{k-1}$ which substantially facilitates the handling of the difference equations of the type (4.1-1). The form of the polynomials is:

$$A(z) = 1 + a_1 z^{-1} + \dots + a_n z^{-n} = 1 + \tilde{A}(z)$$

$$B(z) = b_o + b_1 z^{-1} + \dots + b_n z^{-n}$$

$$C(z) = 1 + c_1 z^{-1} + \dots + c_n z^{-n}$$

$$D(z) = 1 + d_1 z^{-1} + \dots + d_n z^{-n} \qquad (4.1\text{-}4)$$

where a common n order was used, allowing of course for the particular coefficients being zeros within the frame of physical realizability. This latter condition means that the order of the denominators should be greater than or equal with that of the numerator.

ÅSTRÖM and his co-workers have later on pointed out that practical viewpoints do not recommend distinction between the polynomials $A(z)$ and $D(z)$ as the reduction to a common denominator can always be carried out and the form of our model takes a simpler form as Fig. 4.1-2 shows. (The redundant poles and zeros, introduced by the reduction to a

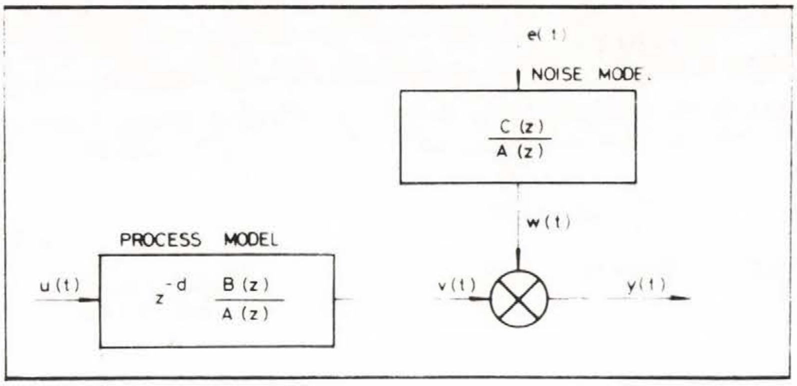

Fig. 4.1-2

common denominator, can be eliminated by relatively simple processes.) In this case $D(z) = A(z)$ and the system equation will take the form

$$y_k = z^{-d} \frac{B(z)}{A(z)} u_k + \frac{C(z)}{A(z)} e_k = v_k + w_k \tag{4.1-5}$$

It can be seen on the right side of the relations (4.1-3) and (4.1-5) that the process model and noise model are separated.

The methods suggested by various authors virtually apply models which can be considered as subcases of (4.1-3). The following table considers the most important ones of these.

Table 4.1-I

Sign	$C(z)/D(z)$	Author	Output noise	Equation error
MLG	$C(z)/D(z)$	ÅSTRÖM	ARMA	ARMA
ML	$C(z)/A(z)$	ÅSTRÖM	ARMA	MA
LS	$1/A(z)$	KALMAN	AR	white
SGLS	1	STEIGLITZ	white	MA
GLS	$1/A(z).H(z)$	CLARKE	AR	AR
IV	$C(z)/A(z).H(z)$	TALMON	ARMA	ARMA

Abbreviations used

MLG = Maximum Likelihood Generalized
ML = Maximum Likelihood
LS = Least Squares
SGLS = Steiglitz type Generalized Least Squares
GLS = Generalized Least Squares
IV = Instrumental Variable

The equation error in the system (in the equations), linear in parameters, denotes the additive error term, the noise.

4.2 Optimization of the sampling period in the case of white output noise

On the basis of Table 4.1-I, the case of the white output noise is denoted by the Eq. $C(z) = A(z)$. Consider as a study object the first order system

$$y_k = \frac{b_1 z^{-1}}{1 + a_1 z^{-1}} u_k + e_k \qquad (4.2\text{-}1)$$

whose discrete parameters are the coefficients b_1 and a_1. Determine on the basis of Eq. (4-17) the information matrix relative to the estimation of the coefficients. For the calculations we need the residual

$$\varepsilon_k = y_k - \frac{b_1 z^{-1}}{1 + a_1 z^{-1}} u_k \qquad (4.2\text{-}2)$$

obtainable from (4.2-1), as now $\underline{p} = [a_1, b_1]^T$ and thus

$$\underline{\underline{J}}_N = \frac{N}{\lambda^2} \begin{bmatrix} E\left\{\left[\frac{\partial \varepsilon_k}{\partial a_1}\right]^2\right\} & E\left\{\frac{\partial \varepsilon_k}{\partial a_1} \frac{\partial \varepsilon_k}{\partial b_1}\right\} \\ E\left\{\frac{\partial \varepsilon_k}{\partial b_1} \frac{\partial \varepsilon_k}{\partial a_1}\right\} & E\left\{\left[\frac{\partial \varepsilon_k}{\partial b_1}\right]^2\right\} \end{bmatrix} \qquad (4.2\text{-}3)$$

Remember that N denotes the number of the processed samples, λ the standard deviation of the source noise. The necessary partial derivatives in our case are

$$\frac{\partial \varepsilon_k}{\partial a_1} = \frac{b_1 z^{-2}}{(1 + a_1 z^{-1})^2} u_k \qquad (4.2\text{-}4)$$

and

$$\frac{\partial \varepsilon_k}{\partial b_1} = - \frac{z^{-1}}{(1 + a_1 z^{-1})} u_k \,. \qquad (4.2\text{-}5)$$

The PARSEVAL theorem can be used to calculate the expected values being in the information matrix. The relations are simple if the input signal is assumed to be, too, a white noise alike the output noise. As the effect of the power level of the input signal is not under investigation so its variance is considered as a unit. The latter two conditions involve, therefore, the existence of the relations

$$E\{u_k^2\} = 1 \quad \text{and} \quad E\{u_k u_m\} = 0, \quad \text{if} \quad m \neq k\,. \qquad (4.2\text{-}6)$$

The expected values in the particular elements of the information matrix are by turn

$$E\left\{\left[\frac{\partial \varepsilon_k}{\partial a_1}\right]^2\right\} = \frac{1}{2\pi j} \oint \frac{b_1 z^{-2}}{(1+a_1 z^{-1})^2} \cdot \frac{b_1 z^2}{(1+a_1 z)^2} \frac{dz}{z} = \frac{b_1^2(1+a_1^2)}{(1-a_1^2)^3}$$

$$(4.2\text{-}7)$$

$$E\left\{\frac{\partial \varepsilon_k}{\partial a_1} \cdot \frac{\partial \varepsilon_k}{\partial b_1}\right\} = \frac{-1}{2\pi j} \oint \frac{z^{-1}}{(1+a_1 z^{-1})} \frac{b_1 z^2}{(1+a_1 z)^2} \frac{dz}{z} =$$

$$= \frac{a_1 b_1}{(1-a_1^2)^2} \tag{4.2-8}$$

$$E\left\{\left[\frac{\partial \varepsilon_k}{\partial b_1}\right]^2\right\} = \frac{1}{2\pi j} \oint \frac{z^{-1}}{(1+a_1 z^{-1})} \frac{z}{(1+a_1 z)} \frac{dz}{z} = \frac{1}{1-a_1^2} . \tag{4.2-9}$$

Appendix 1. contains the calculations in detail. The information matrix obtained as a result is

$$\underline{\underline{J}}_N = \frac{N}{\lambda^2} \begin{bmatrix} \dfrac{b_1^2 (1+a_1^2)}{(1-a_1^2)^3} & \dfrac{a_1 b_1}{(1-a_1^2)^2} \\ \\ \dfrac{a_1 b_1}{(1-a_1^2)^2} & \dfrac{1}{1-a_1^2} \end{bmatrix} \tag{4.2-10}$$

and its inverse (after simple calculations):

$$\underline{\underline{J}}_N^{-1} = \frac{\lambda^2}{N} \begin{bmatrix} \dfrac{(1-a_1^2)^3}{b_1^2} & -\dfrac{a_1(1-a_1^2)^2}{b_1} \\ \\ -\dfrac{a_1(1-a_1^2)^2}{b_1} & (1-a_1^2)(1+a_1^2) \end{bmatrix} . \tag{4.2-11}$$

As already mentioned, in optimizing the sampling period the improvement of the parameter estimation of the continuous system is intended. By using a step response equivalent transformation, we obtain a continuous system corresponding to the investigated first-order discrete system, with the following transfer function according to (2.9)

$$H(s) = \frac{\beta_1}{s+\alpha_1} = \frac{K}{1 + sT} \tag{4.2-12}$$

where the relations again:

$$\alpha_1 = -\frac{1}{h}\ln(-a_1); \quad \beta_1 = -\frac{b_1 \ln(-a_1)}{h(1 + a_1)} = \frac{b_1\,\alpha_1}{1 + a_1}$$

$$K = \frac{\beta_1}{\alpha_1} = \frac{b_1}{1+a_1}; \quad T = \frac{1}{\alpha_1} = -\frac{h}{\ln(-a_1)}. \tag{4.2-13}$$

The parameters of the continuous system are thus calculated through complicated nonlinear function connections from the parameters b_1 and a_1 of the discrete system. The uncertainty, the standard deviation (or the information boundary related to these) in the parameters b_1 and a_1 extends in a very complicated form to the parameter β_1 and α_1. The accurate consideration of this error extension is a fairly complicated task. But consider the transformation

$$\hat{\underline{p}}_f = [\hat{\alpha}_1, \hat{\beta}_1]^T = \hat{\underline{p}}_f(\hat{\underline{p}}_d) = \hat{\underline{p}}_f(\hat{\underline{p}}_d = [\hat{a}_1, \hat{b}_1]^T) \tag{4.2-14}$$

and its linear approximation for a given working point, $\underline{p}_c$ i.e.

$$\hat{\underline{p}}_c \cong \underline{p}_c + \frac{d\,\hat{\underline{p}}_c}{d\,\hat{\underline{p}}_d^T}\left(\hat{\underline{p}}_d - \underline{p}_d\right) = \underline{p}_c + \frac{d\,\hat{\underline{p}}_c}{d\,\hat{\underline{p}}_d^T}\;\Delta\,\hat{\underline{p}}_d \,. \qquad (4.2\text{-}15)$$

(Here the signal ^ refers to the estimated values, the subscripts c and d denote the parameter vectors of the continuous and discrete transfer functions.) With the help of the JACOBIAN matrix in (4.2-15),

$$\underline{\underline{A}}_{\alpha\beta} = \frac{d\,\hat{\underline{p}}_c}{d\,\hat{\underline{p}}_d^T} = \begin{bmatrix} \dfrac{\partial\alpha_1}{\partial a_1} & \dfrac{\partial\alpha_1}{\partial b_1} \\[2ex] \dfrac{\partial\beta_1}{\partial a_1} & \dfrac{\partial\beta_1}{\partial b_1} \end{bmatrix} \qquad (4.2\text{-}16)$$

we can transform the covariance matrix of the estimated parameter vector $\hat{\underline{p}}_d$:

$$\underline{\underline{K}}_{f_1} = \underline{\underline{A}}_{\alpha\beta}\,\underline{\underline{K}}_d\,\underline{\underline{A}}_{\alpha\beta}^T \,. \qquad (4.2\text{-}17)$$

in the way known in mathematical statistics [59], and its lower bound

$$\underline{\underline{J}}_{\alpha\beta}^{-1} = \underline{\underline{A}}_{\alpha\beta}\;\underline{\underline{J}}_d^{-1}\;\underline{\underline{A}}_{\alpha\cdot\beta}^T \,. \qquad (4.2\text{-}18)$$

With $\underline{\underline{J}}^{-1}$ we have made appear subscripts α and β, too, because it refers to the parameters α,β of the continuous system. The matrix $\underline{\underline{J}}_{K,T}^{-1}$ referring to parameters K,T can also be determined alike (4.2-18):

$$\underline{\underline{J}}_{KT}^{-1} = \underline{\underline{A}}_{KT} \ \underline{\underline{J}}_d^{-1} \ \underline{\underline{A}}_{KT}^{T}, \tag{4.2-19}$$

where

$$\underline{\underline{A}}_{KT} = \begin{bmatrix} \dfrac{\partial K}{\partial a_1} & \dfrac{\partial K}{\partial b_1} \\ \\ \dfrac{\partial T}{\partial a_1} & \dfrac{\partial T}{\partial b_1} \end{bmatrix}. \tag{4.2-20}$$

On the basis of the detailed calculations in Appendix 2.

$$\underline{\underline{A}}_{\alpha\beta} = \begin{bmatrix} -\dfrac{1}{a_1 \ h} & 0 \\ \\ \dfrac{-b_1}{h}\left[\dfrac{1+a_1-a_1 \ \ln(-a_1)}{a_1(1+a_1)^2}\right] & -\dfrac{\ln(-a_1)}{h(1+a_1)} \end{bmatrix} \tag{4.2-21}$$

$$\underline{\underline{A}}_{KT} = \begin{bmatrix} -\dfrac{b_1}{(1+a_1)^2} & \dfrac{1}{1+a_1} \\ \\ \dfrac{h}{a_1 \ \ln^2(-a_1)} & 0 \end{bmatrix}. \tag{4.2-22}$$

In Section A.2 we have elaborated also the calculation of the CRAMER-RAO lower bound which resulted in

$$\underline{\underline{J}}^{-1}_{\alpha\beta} = \frac{\lambda^2}{NT^2x^2}\begin{bmatrix} \dfrac{(1+e^{-x})^3(1-e^{-x})}{K^2e^{-2x}} & \dfrac{\left[1-e^{-x}(x+e^{-x})\right](1+e^{-x})}{Ke^{-2x}} \\ \\ \dfrac{\left[1-e^{x}(x+e^{-x})\right](1+e^{-x})}{Ke^{-2x}} & \dfrac{(1+e^{-x})\{\left[1-e^{-x}(x+e^{-x})\right]^2+x^2e^{-2x}\}}{(1-e^{-x})\,e^{-2x}} \end{bmatrix} \tag{4.2-23}$$

or

$$\underline{\underline{J}}^{-1}_{KT} = \frac{\lambda^2}{N}\begin{bmatrix} \dfrac{2(1+e^{-x})}{(1-e^{-x})} & \dfrac{T(1+e^{-x})\left[1-e^{-2x}(1+e^{-x})\right]}{K\,x(1-e^{x})\,e^{-x}} \\ \\ \dfrac{T(1+e^{-x})\left[1-e^{-2x}(1+e^{-x})\right]}{Kx(1-e^{-x})\,e^{-x}} & \dfrac{T^2(1+e^{-x})^3\,(1-e^{-x})}{K^2x^2\,e^{-2x}} \end{bmatrix} \tag{4.2-24}$$

where the relative sampling rate $x = h/T$ has been introduced.

According to the inequality CRAMER-RAO $J^{-1}_{1,1}$ denotes the theoretical lower bound of the variance of α and J^{-1}_{11} that of K. Similarly $J^{-1}_{2,2}$ denotes the theoretical lower bound of the variance of β and J^{-1}_{22} denotes that of T.

The CRAMER-RAO theorem is related to N number of processed samples. As the samples are obtained by sampling the time functions in h sampling periods, two fundamental cases can be distinguished. In one case the number (N) of the processed samples, in the other the observation time

$$T_m = N \cdot h \tag{4.2-25}$$

is fixed.

In both cases the development of the inverse of the information matrix is investigated as a function of h (or $x = h/T$). On the basis of (4.2-23), the lower bound of the variance of $\hat{\alpha}$ is

$$\mathrm{var}(\hat{\alpha}) \geqq \frac{\lambda^2}{N} \quad \frac{(1+e^{-x})^3(1-e^{-x})}{T^2K^2x^2\ e^{-2x}} = \frac{\lambda^2}{N} J^{-1}_{1,1}(x) \tag{4.2-26}$$

whence

$$J^{-1}_{1,1}(x)\Big|_{N=\mathrm{const}} = \frac{(1+e^{-x})^3\ (1-e^{-x})}{T^2K^2x^2\ \ e^{-2x}} \leqq \frac{N\ \mathrm{var}(\hat{\alpha})}{\lambda^2} \tag{4.2-27}$$

$$J^{-1}_{1,1}(x)\Big|_{T_m=\mathrm{const}} = \frac{(1+e^{-x})^3\ (1-e^{-x})}{T\ K^2\ \ x\ e^{-2x}} \leqq \frac{T_m\ \mathrm{var}(\hat{\alpha})}{2} \ . \tag{4.2-28}$$

Here in J^{-1}_{11} the occurrence of T^2 and K^2, as they are system parameters, changes only the scaling, but not the location of the occasional optimum in x. The lower bound of the variance of $\hat{\beta}$

$$\mathrm{var}(\hat{\beta}) \geqq \frac{\lambda^2}{N} \frac{(1+e^{-x}) \{[1-e^{-x}(x+e^{-x})]^2 + x^2 e^{-2x}\}}{T^2 x^2 (1 - e^{-x}) e^{-2x}} = \frac{\lambda^2}{N} J^{-1}_{2,2}(x) \tag{4.2-29}$$

whence

$$J^{-1}_{2,2}(x)\Big|_{N=\mathrm{const}} = \frac{(1+e^{-x})\{[1-e^{-x}(x+e^{-x})]^2 + x^2 e^{-2x}\}}{T^2 x^2 (1-e^{-x}) e^{-2x}}$$

$$\leqq \frac{N}{\lambda^2} \mathrm{var}(\hat{\beta}), \tag{4.2-30}$$

and

$$J^{-1}_{2,2}(x)\Big|_{T_m=\mathrm{const}} = \frac{(1+e^{-x})\{[1-e^{-x}(x+e^{-x})]^2 + x^2 e^{-2x}\}}{T\, x\, (1-e^{-x})\, e^{-2x}}$$

$$\leqq \frac{T_m}{\lambda^2} \mathrm{var}(\hat{\beta}). \tag{4.2-31}$$

Investigate only var $(\hat{K})$ from (4.2-24), as the behaviour of $\mathrm{var}(\hat{T})$ can be judged also from $\mathrm{var}(\hat{\alpha})$.

$$\mathrm{var}(\hat{K}) \geqq \frac{\lambda^2}{N} \frac{2(1+e^{-x})}{(1-e^{-x})} = \frac{\lambda^2}{N} J^{-1}_{11}(x) \tag{4.2-32}$$

whence

$$J^{-1}_{11}(x)\Big|_{N=\mathrm{const}} = \frac{2(1+e^{-x})}{1-e^{-x}} \leqq \frac{N}{\lambda^2} \mathrm{var}(\hat{K}) \tag{4.2-33}$$

and

$$J_{11}^{-1}(x)\Big|_{T_m=\text{const}} = \frac{T\,x\,(1+e^{-x})}{1-e^{-x}} \leqq \frac{T_m}{\lambda^2}\,\mathrm{var}(\hat{K}). \tag{4.2-34}$$

Fig. 4.2-1 shows the normalized CRAMER-RAO lower bounds figuring in (4.2-27), (4.2-28), (4.2-30), (4.2-31), (4.2-33) and (4.2-34) in the function of x. On the left side of the figure N=const, on the right side, however, T_m = const.

It is interesting to note that there is a definite minimum of the variances $\hat{\alpha}$ and $\hat{\beta}$ in case N = constant at the relative sampling rate $x_{opt} \approx 1$, while in the same case a sampling period as long as possible (steady state) would be optimal for the estimation of the gain $\hat{K}$. On the other hand, in the case of T_m = constant, the minimum of the variance of every parameter can be obtained at the value $x_{opt} \approx 0$. This latter phenomenon is relatively easy to explain, being obvious that from an available recorded signal the most information can be obtained by sampling it as often as possible. The figure shows furthermore that with the values $x < 1$ there is no more essential decrease in the variance of the parameters so that also in this case the choice $x_{opt} \approx 1$ can be accepted as minimum. With respect to the joint estimation properties of the parameters, the investigation of the trace and determinant of the matrices was carried out, too. The investigated quantities:

$$\mathrm{tr}\left(\underline{\underline{J}}_{\alpha\beta}^{-1}\right) = \frac{\lambda^2(1+e^{-x})}{NT^2x^2\,e^{-2x}}\left[\frac{(1+e^{-x})^2(1-e^{-x})}{K^2} + \frac{\left[1-e^{-x}\,(x+e^{-x})\right]^2 + x^2e^{-2x}}{(1-e^{-x})}\right] \tag{4.2-35}$$

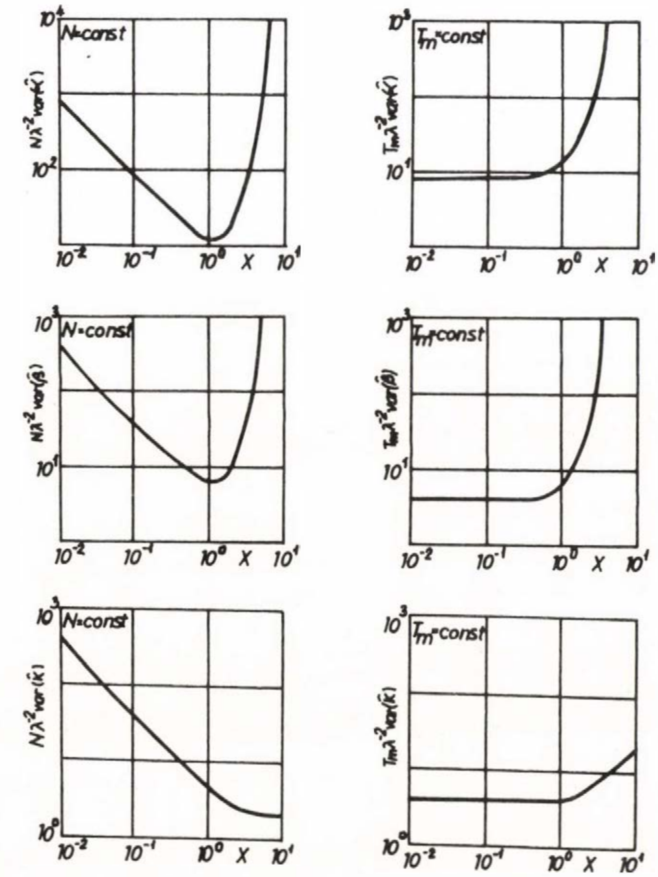

Fig. 4.2-1

$$\left|\underline{\underline{J}}^{-1}_{\alpha\beta}\right| = \frac{\lambda^4 (1+e^{-x})^2}{N^2 T^4 \; x^4 e^{-4x}} \{ \left[1-e^{-x}(x-e^{-x})\right]^2 \left[(1+e^{-x})^2 -1\right] +$$

$$+ (1+e^{-x})^2 \; x^2 \; e^{-2x} \} \qquad (4.2\text{-}36)$$

and

$$\mathrm{tr}\left(\underline{\underline{J}}^{-1}_{KT}\right) = \frac{\lambda^2}{N} \left[\frac{2(1+e^{-x})}{(1-e^{-x})} + \frac{T^2(1+e^{-x})^3 (1-e^{-x})}{K^2 x^2 \; e^{-2x}} \right] \qquad (4.2\text{-}37)$$

$$\left|\underline{\underline{J}}^{-1}_{KT}\right| = \frac{\lambda^4 \; T^2 (1+e^{-x})^2}{N^2 K^2 \; (1-e^{-x})^2 \; x^2 e^{-2x}} \{ 2(1-e^{-2x})^2 - \left[1-e^{-2x}(1+e^{-x})\right]\} .$$

$$(4.2\text{-}38)$$

In Fig. 4.2-2 the quantitiy $N \lambda^{-2} \; \mathrm{tr}\left(\underline{\underline{J}}^{-1}_{\alpha\beta}\right)$ was presented in the function of x for the case N = constant, with parameters in K. It is easy to see that the location of the minimum - practically independently from K - is in the environment of $x_{opt} \approx 1$.

In Fig. 4.2-3 the quantity $T_m \lambda^{-2} \; \mathrm{tr}\left(\underline{\underline{J}}^{-1}_{\alpha\beta}\right)$ is presented according to the case T_m = const. in the function of x and parametrizing by K. The fact experienced while investigating the variances of the parameters was again arrived at: in case of a given observation time the shortest sampling time $(x_{opt} \approx 0)$ is the best.

Fig. 4.2-4 shows the quantity $N^2 \; \lambda^{-4} \; \left|\underline{\underline{J}}^{-1}_{\alpha\beta}\right|$ and Fig. 4.2-5 the quantity $T_m^{\;2} \; \lambda^{-4} \left|\underline{\underline{J}}^{-1}_{\alpha\beta}\right|$ in the function of x (for K = 1 as it changes only the scaling) according to the cases N = const. and T_m = const.

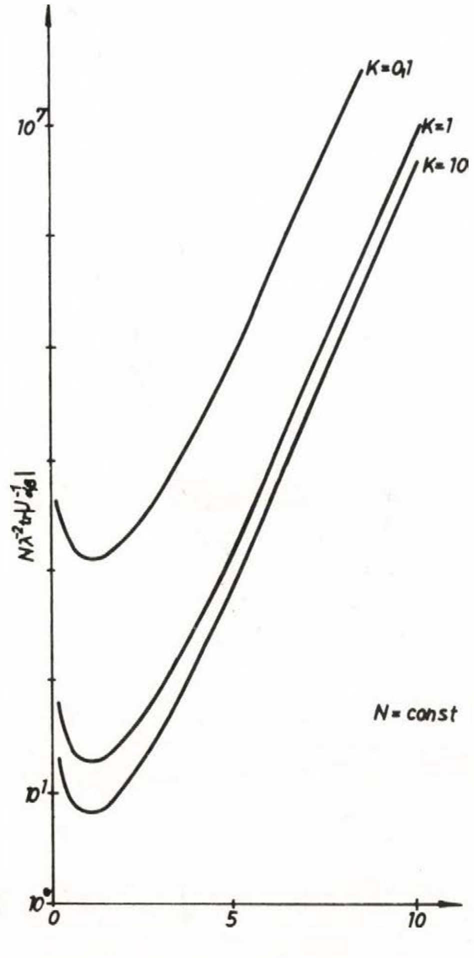

Fig. 4.2-2

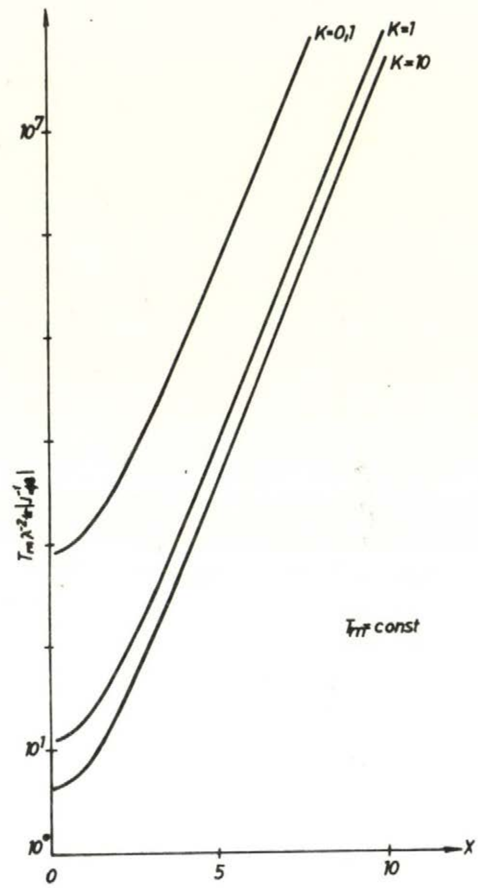

Fig. 4.2-3

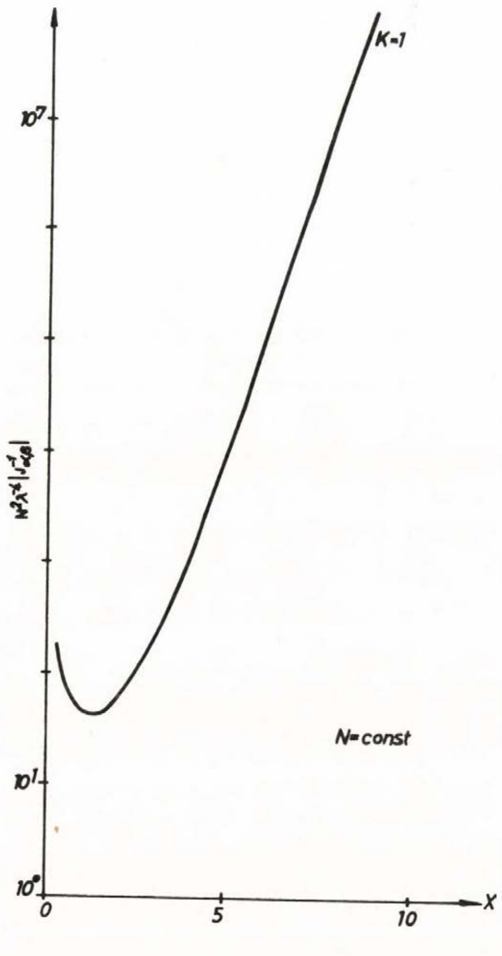

Fig. 4.2-4

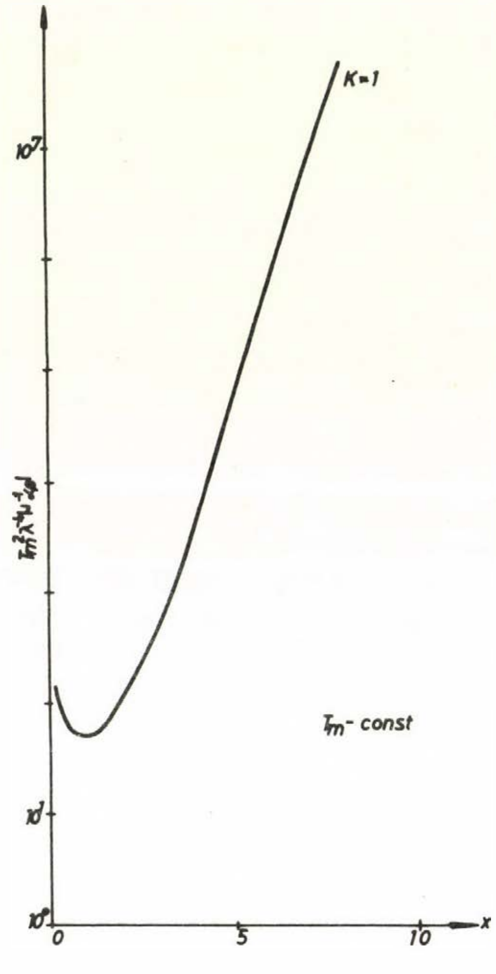

Fig. 4.2-5

In Fig. 4.2-4 we see again the optimal sampling time $x_{opt} \approx 1$ but interestingly Fig. 4.2-5 does not point at the value $x_{opt} \approx 0$ but there is a definite optimum in the previous domain.

Fig. 4.2-6 shows the dependence on x parametrizing in K of the $N\lambda^{-2}\,\mathrm{tr}\left(\underline{\underline{J}}_{KT}^{-1}\right)$, Fig. 4.2-7 that of the quantity $T_m\,\lambda^{-2}\,\mathrm{tr}\left(\underline{\underline{J}}_{KT}^{-1}\right)$.

For the case N = const $x_{opt} \approx 1$ dependent on K is obtained while for the case T_m = const. the $x_{opt} \approx 0$ is the optimal.

Fig. 4.2-8 shows the quantity $N^2\lambda^{-4}\left|\underline{\underline{J}}_{KT}^{-1}\right|$ and the quantity $T_m^2\,\lambda^{-4}\left|\underline{\underline{J}}_{KT}^{-1}\right|$. Both cases refer to the optimal sampling time $x_{opt} \sim \infty$.

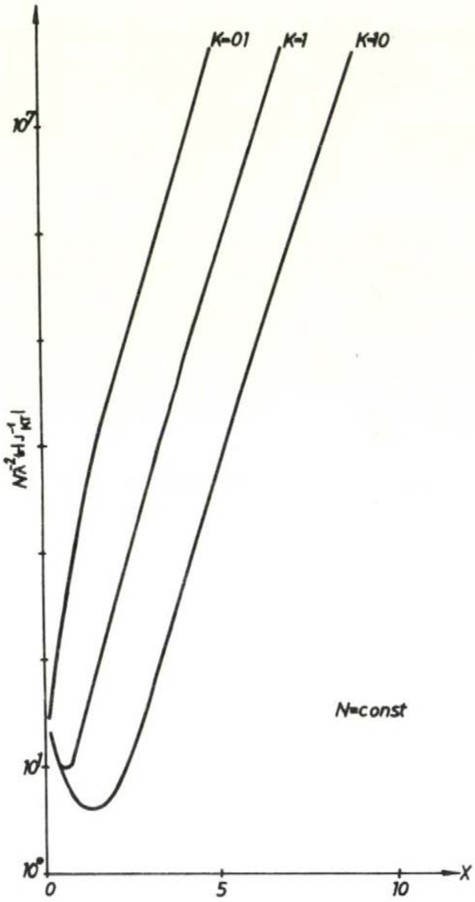

Fig. 4.2-6

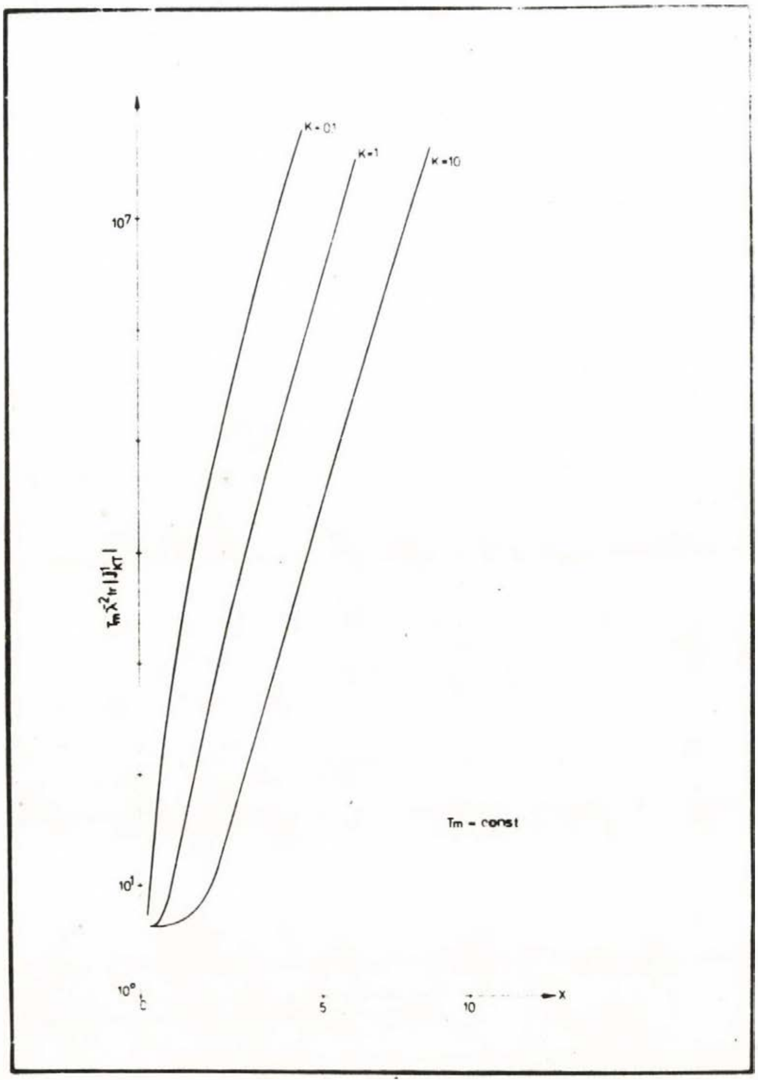

Fig. 4.2-7

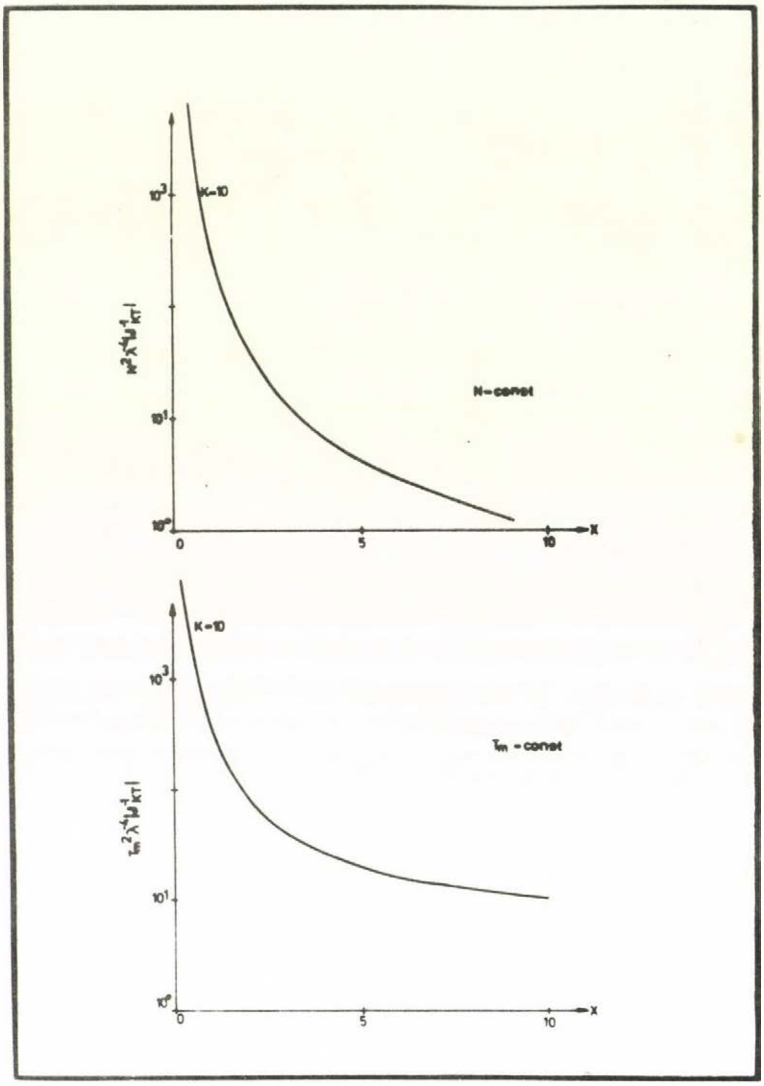

Fig. 4.2-8

APPENDIX

Appendix 1.

The evaluation of the integral $\frac{1}{2\pi j}\oint F(z)\, z^{n-1}$ can be carried out e.g. with the residue theorem according to which

$$\frac{1}{2\pi j}\oint_{\Gamma} F(z)\, z^{n-1}\, dz = \sum_{i=1}^{R} \operatorname*{Res}_{z=p} F(z)\, z^{n-1} \qquad (A.1\text{-}1)$$

where the summation must be extended to all p_i poles of $F(z)z^{n-1}$ within the curve Γ of the complex plane z. (Here R denotes the number of residues). As we consider only a stable system, in our case Γ will obviously denote the unit circle. This means that from the roots of the integrands of the Eq. (4.2-7) - (4.2-9), the root $z = \frac{1}{a}$ is located outside the unit circle and we do not consider it. Because of the condition of stability, we have to avail ourselves of the condition $a < 1$. Thus

$$\frac{1}{2\pi j}\oint \frac{b\, z^{-2}}{(1+az^{-1})^{2}} \cdot \frac{b\, z^{2}}{(1+az)^{2}}\,\frac{dz}{z} =$$

$$= \operatorname*{Res}_{z=-a} \frac{b^{2} z}{(z+a)^{2}(1+az)^{2}} = \frac{d}{dz}\,\frac{(z+a)^{2}\, b^{2} z}{(z+a)^{2}\,(1+az)^{2}}\Bigg|_{z=-a} =$$

$$= \frac{b^{2}(1+az) - 2a(1+az)\, b^{2} z}{(1+az)^{4}}\Bigg|_{z=-a} =$$

$$= \frac{b^{2}\left[(1+az) - 2az\right]}{(1+az)^{3}}\Bigg|_{z=-a} = \frac{b^{2}(1+a^{2})}{(1-a^{2})^{3}}\,. \qquad (A.1\text{-}2)$$

Likewise

$$\frac{1}{2\pi j} \oint \frac{z^{-1}}{(1 + az^{-1})} \quad \frac{bz^2}{(1+az)^2} \; \frac{dz}{z} =$$

$$= \operatorname*{Res}_{z=-a} \quad \frac{1}{(z+a)} \quad \frac{bz}{(1+az)} = \frac{-ba}{(1-a^2)^2} \qquad \text{(A.1-3)}$$

and

$$\frac{1}{2\pi j} \oint \frac{z^{-1}}{(1+az^{-1})} \quad \frac{1}{(1+az)} \; \frac{dz}{z} = \operatorname*{Res}_{z=-a} \; \frac{1}{(z+a)} \quad \frac{1}{(1 + az)} =$$

$$= (z+a) \; \frac{1}{(z+a)(1+az)} \Bigg|_{z=-a} = \frac{1}{(1-a^2)}$$

Appendix 2.

On the basis of the relations (4.2-13), the partial derivatives making the elements of $\underline{\underline{A}}_{\alpha\beta}$ and $\underline{\underline{A}}_{KT}$ are

$$\frac{\partial \alpha_1}{\partial a_1} = - \frac{1}{h\, a_1} , \qquad \frac{\partial \alpha_1}{\partial b_1} = 0$$

$$\frac{\partial \beta_1}{\partial a_1} = - \frac{b_1}{h} \left[\frac{1+a_1-a_1 \ln(-a_1)}{a_1(1+a_1)^2} \right] ;$$

$$\frac{\partial \beta_1}{\partial b_1} = - \frac{\ln(-a_1)}{h(1+a_1)} \tag{A.2-1}$$

as well as

$$\frac{\partial K}{\partial a_1} = \frac{-b_2}{(1+a_1)^2} ; \quad \frac{\partial K}{\partial b_1} = \frac{1}{1+a_1}$$

$$\frac{\partial T}{\partial a_1} = \frac{h}{a_1 \ln^2 (-a_1)} ; \quad \frac{\partial T}{\partial b_1} = 0 \tag{A.2-2}$$

on the basis of which (4.2-21) and (4.2-22) can be constructed.

Determine the matrix

$$\underline{\underline{J}}_{KT}^{-1} = \underline{\underline{A}}_{KT} \ \underline{\underline{J}}_{d}^{-1} \ \underline{\underline{A}}_{KT}^{T} = \frac{\lambda^2}{N} \begin{bmatrix} J_{11}^{-1} & J_{12}^{-1} \\ J_{21}^{-1} & J_{22}^{-1} \end{bmatrix} \tag{A.2-3}$$

where the upper index in J_{ij}^{-1} indicates only the inverse property of the matrix.

By applying the rules of matrix multiplication

$$J_{11}^{-1} = \frac{b_1^2}{(1+a_1)^4} \quad \frac{(1-a_1^2)^3}{b_1^2} - \frac{2b_1 a_1 (1-a_1^2)^2}{b_1(1+a_1)(1+a_1)^2} +$$

$$+ \frac{(1-a_1^2)(1+a_1^2)}{(1+a_1)^2} = \frac{2(1-a_1)}{(1+a_1)} = \frac{2(1+e^{-\alpha h})}{(1-e^{-\alpha h})} \qquad \text{(A.2-4)}$$

where the relations (4.2-13) have been considered.

By introducing the relative sampling rate $x = h/T$

$$J_{11}^{-1} = \frac{2(1+e^{-x})}{(1-e^{-x})} \,. \qquad \text{(A.2-5)}$$

Proceeding likewise

$$J_{12}^{-1} = J_{21}^{-1} = \frac{h}{a_1 \ln^2(-a_1)} \quad \frac{(1-a_1^2)^3}{b_1^2} \quad \frac{(-b_1)}{(1+a_1)^2} +$$

$$+ \frac{h}{a_1 \ln^2(-a_1)} \quad \frac{(-a_1)(1-a_1^2)^2}{b_1 \, (1+a_1)} =$$

$$= \frac{-h}{a_1 \ln^2(-a_1)} \quad \frac{(1-a_1)}{b_1} \left[1-a_1^2 \, (1-a_1) \right] \qquad \text{(A.2-6)}$$

or with the help of (4.2-13)

$$J_{12}^{-1} = J_{21}^{-1} = \frac{1}{h\alpha\beta} \frac{(1+e^{-\alpha h})}{(1-e^{-\alpha h})} \frac{1-e^{-2\alpha h}(1+e^{-\alpha h})}{e^{-\alpha h}} =$$

$$= \frac{T(1+e^{-x})\left[1-e^{-2x}\ (1+e^{-x})\right]}{Kx\ (1-\ e^{-x})\ e^{-x}} \ . \tag{A.2-7}$$

Finally

$$J_{22}^{-1} = \frac{h^2}{a_1^2\ \ln^4(-a_1)} \quad \frac{(1-a_1^2)}{(b_1^2)} \tag{A.2-8}$$

whence

$$J_{22}^{-1} = \frac{1}{(\alpha h)^2\ \beta^2} \frac{(1+e^{-\alpha h})^{\ 3}\ (1-e^{-\alpha h})}{e^{-2\alpha h}} =$$

$$= \frac{T^2}{K^2 x^2} \quad \frac{(1+e^{-x})^3\ (1-e^{-x})}{e^{-2x}} \ . \tag{A.2-9}$$

Determine now the element of the matrix

$$\underline{\underline{J}}_{\alpha\beta}^{-1} = \underline{\underline{A}}_{\alpha\beta}\ \underline{\underline{J}}_{d}^{-1}\ \underline{\underline{A}}_{\alpha\beta}^{T} = \frac{\lambda^2}{N} \begin{bmatrix} J_{1,1}^{-1} & J_{1,2}^{-1} \\ J_{2,1}^{-1} & J_{2,2}^{-1} \end{bmatrix} \tag{A.2-10}$$

where in $\underline{J}^{-1}_{i,j}$ the upper index points again at the inverse, furthermore we have separated the subscripts by a comma. By carrying out the calculations by elements:

$$J^{-1}_{1,1} = \frac{1}{K^2h^2}\,\frac{(1-a_1^2)^3}{b_1^2} = \frac{1}{h^2}\,\frac{\alpha^2}{\beta^2}\,\frac{(1+e^{-\alpha h})^3\,(1-e^{-\alpha h})}{e^{-2\alpha h}} =$$

$$= \frac{1}{T^2K^2x^2}\,\frac{(1+e^{-x})^3(1-e^{-x})}{e^{-2x}}\;; \qquad (A.2\text{-}11)$$

$$J^{-1}_{1,2} = J^{-1}_{2,1} = \frac{(1-a_1^2)^3}{b_1^2}\,\frac{(-b_1)}{h}\,\frac{\left[1+a_1-a_1\ln(-a_1)\right]}{a_1(1+a_1)^2}\,\frac{(-1)}{a_1h} +$$

$$+ \frac{(-a_1)\;(1-a_1^2)^2}{b_1}\,\frac{-\ln(-a_1)}{h(1+a_1)}\,\frac{(-1)}{a_1h} =$$

$$= \frac{(1-a_1)^2\;(1+a_1)}{a_1^2\;\;b_1h}\left[1-a_1\ln(-a_1)-a_1^2\right] =$$

$$= \frac{1}{(\alpha h)^2}\,\frac{\alpha^3}{T^2K}\,\frac{(1+e^{-\alpha h})^2\left[1-e^{-\alpha h}\,(\alpha h+e^{-\alpha h})\right]}{e^{-2\alpha h}}$$

$$= \frac{1}{T^2Kx^2}\,\frac{(1+e^{-x})\left[1-e^{-x}\,(x+e^{-x})\right]}{e^{-2x}}\;; \qquad (A.2\text{-}12)$$

$$J^{-1}_{2,2} = \frac{(1-a_1^2)^{3}}{b_1^2} \left\{ \frac{-b_1}{h} \left[\frac{1+a_1-a_1 \ln(-a_1)}{a_1(1+a_1)^2} \right] \right\}^2 +$$

$$+ 2 \, \frac{-a_1(1-a_1^2)}{b_1} \; \frac{-\ln(-a_1)}{h(1+a_1)} \; \frac{-b_1}{h} \; \frac{1+a_1-a_1 \ln(-a_1)}{a_1(1+a_1)^2} +$$

$$+ \left[\frac{-\ln(-a_1)}{h(1+a_1)} \right]^2 (1-a_1^2)(1+a_1^2) =$$

$$= \frac{(1-a_1)}{h^2 a_1^2 (1+a_1)} \left\{ \left[1-a_1^2 - a_1 \ln(-a_1) \right]^2 + a_1^2 \ln^2(-a_1) \right\} =$$

$$= \frac{(1+e^{-\alpha h})}{h^2(1-e^{-\alpha h})} \; \frac{\left[1-e^{-\alpha h}(\alpha h+e^{-\alpha h}) \right]^2 + e^{-2\alpha h}(\alpha h)^2}{e^{-2\alpha h}} =$$

$$= \frac{(1+e^{-x}) \left\{ \left[1-e^{-x}(x+e^{-x}) \right]^2 + e^{-2x} \, x^2 \right\}}{T^2 x^2 (1-e^{-x}) \; e^{-2x}} \tag{A.2-13}$$

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